Felix Klein's Vision: A School for Mathematical Productivity Regardless of Nationality, Gender, and Area of Research

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Abstract

In 1912, some 330 people donated for a portrait of Felix Klein by the important (Jewish) Impressionist painter Max Liebermann. The list of donors (Fig. 1.1, 1.2, 1.3, and 1.4) includes many foreign mathematicians (many of them were Jewish). Klein supported them regardless of their religion and nationality. Some of their personal connections with Klein go back to his time as a *Privatdozent* in Göttingen. This article considers Klein's impact on foreign mathematicians under the following aspects. Firstly, we look at Klein's aim to create a school of scientific productivity. We briefly compare Klein's way of leading young mathematicians to their own new results with the behaviour of other important German mathematicians in this respect. Secondly, we give an overview of the foreign mathematicians he inspired to write dissertations or make mathematical findings. Some relationships with mathematicians from Italy, Hungary, and Russia are examined in more detail. Thirdly, we investigate what influence Klein's interdisciplinary/joint research seminars had on the results of mathematicians from abroad who participated in these seminars in Göttingen.

1.1 A School for Mathematical Productivity

Felix Klein expressed his vision of founding a school of mathematical productivity early on. Shortly after taking up his first full professorship at the University of Erlangen in 1872 at the age of 23, he wrote to the French mathematician Gaston

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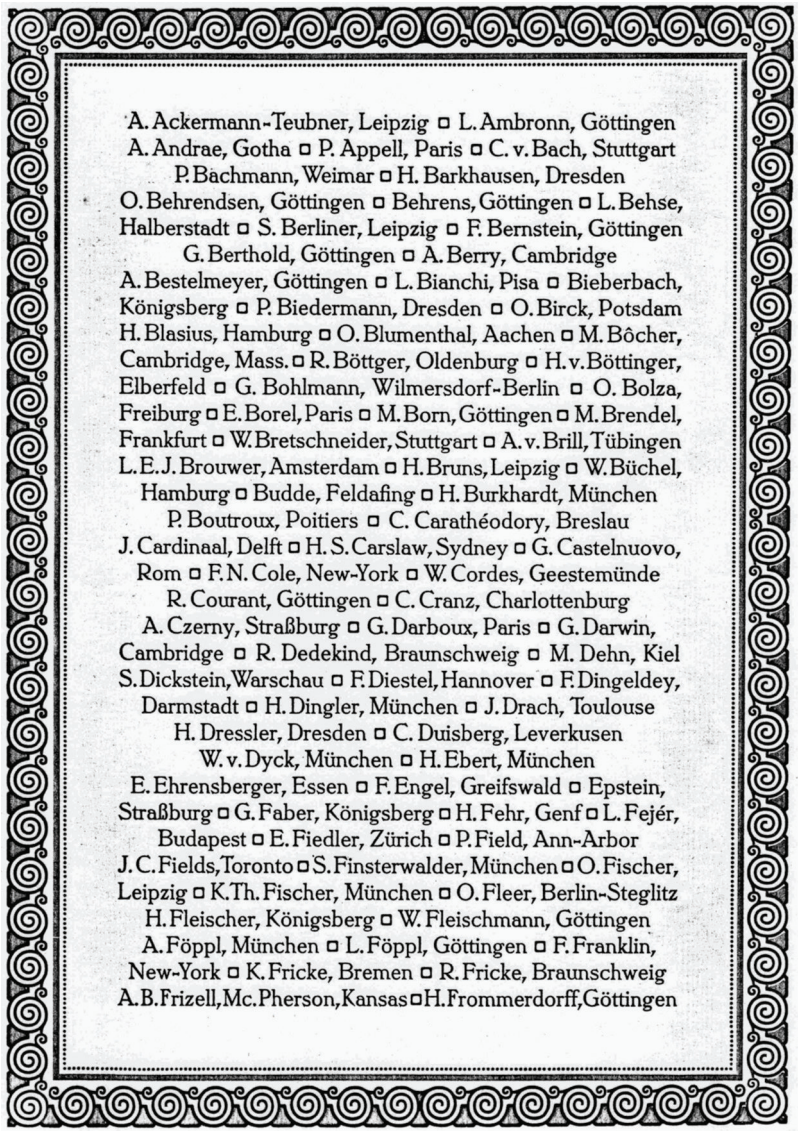
D. Ciesielska et al. (eds.), *Felix Klein's Foreign Students*, Trends in the History of Science, https://doi.org/10.1007/978-3-031-95777-2_1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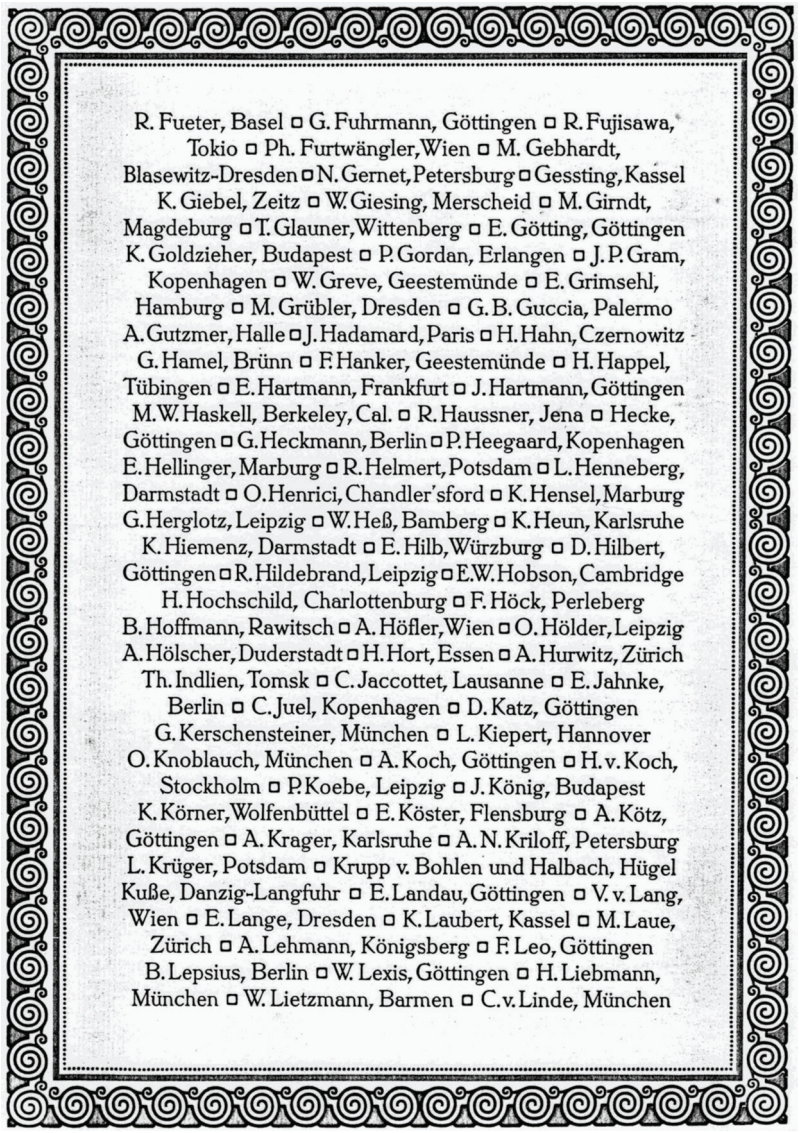
O. Behrendsen, Göttingen ◻ Behrens, Göttingen ◻ L. Behse,
Halberstadt ◻ S. Berliner, Leipzig ◻ F. Bernstein, Göttingen
G. Berthold, Göttingen ◻ A. Berry, Cambridge
A. Bestelmeyer, Göttingen ◻ L. Bianchi, Pisa ◻ Bieberbach,
Königsberg ◻ P. Biedermann, Dresden ◻ O. Birck, Potsdam
H. Blasius, Hamburg ◻ O. Blumenthal, Aachen ◻ M. Bôcher,
Cambridge, Mass. ◻ R. Böttger, Oldenburg ◻ H. v. Böttinger,
Elberfeld ◻ G. Bohlmann, Wilmersdorf-Berlin ◻ O. Bolza,
Freiburg ◻ E. Borel, Paris ◻ M. Born, Göttingen ◻ M. Brendel,
Frankfurt ◻ W. Bretschneider, Stuttgart ◻ A. v. Brill, Tübingen
L. E. J. Brouwer, Amsterdam ◻ H. Bruns, Leipzig ◻ W. Büchel,
Hamburg ◻ Budde, Feldafing ◻ H. Burkhardt, München
P. Boutroux, Poitiers ◻ C. Carathéodory, Breslau
J. Cardinaal, Delft ◻ H. S. Carslaw, Sydney ◻ G. Castelnuovo,
Rom ◻ F. N. Cole, New-York ◻ W. Cordes, Geestemünde
R. Courant, Göttingen ◻ C. Cranz, Charlottenburg
A. Czerny, Straßburg ◻ G. Darboux, Paris ◻ G. Darwin,
Cambridge ◻ R. Dedekind, Braunschweig ◻ M. Dehn, Kiel
S. Dickstein, Warschau ◻ F. Diestel, Hannover ◻ F. Dingeldey,
Darmstadt ◻ H. Dingler, München ◻ J. Drach, Toulouse
H. Dressler, Dresden ◻ C. Duisberg, Leverkusen
W. v. Dyck, München ◻ H. Ebert, München
E. Ehrensberger, Essen ◻ F. Engel, Greifswald ◻ Epstein,
Straßburg ◻ G. Faber, Königsberg ◻ H. Fehr, Genf ◻ L. Fejér,
Budapest ◻ E. Fiedler, Zürich ◻ P. Field, Ann-Arbor
J. C. Fields, Toronto ◻ S. Finsterwalder, München ◻ O. Fischer,
Leipzig ◻ K. Th. Fischer, München ◻ O. Fleer, Berlin-Steglitz
H. Fleischer, Königsberg ◻ W. Fleischmann, Göttingen
A. Föppl, München ◻ L. Föppl, Göttingen ◻ F. Franklin,
New-York ◻ K. Fricke, Bremen ◻ R. Fricke, Braunschweig
A. B. Frizell, Mc. Pherson, Kansas ◻ H. Frommerdorff, Göttingen

Fig. 1.1 A list of donors who sponsored Max Liebermann's painting of Klein's portrait in 1912 [Hillebrand]. H.Frommerdorff = Hermann Trommsdorff, teacher at the *Oberrealschule* in Göttingen (math/physics/chemistry), PhD University of Jena in 1901 [BBF]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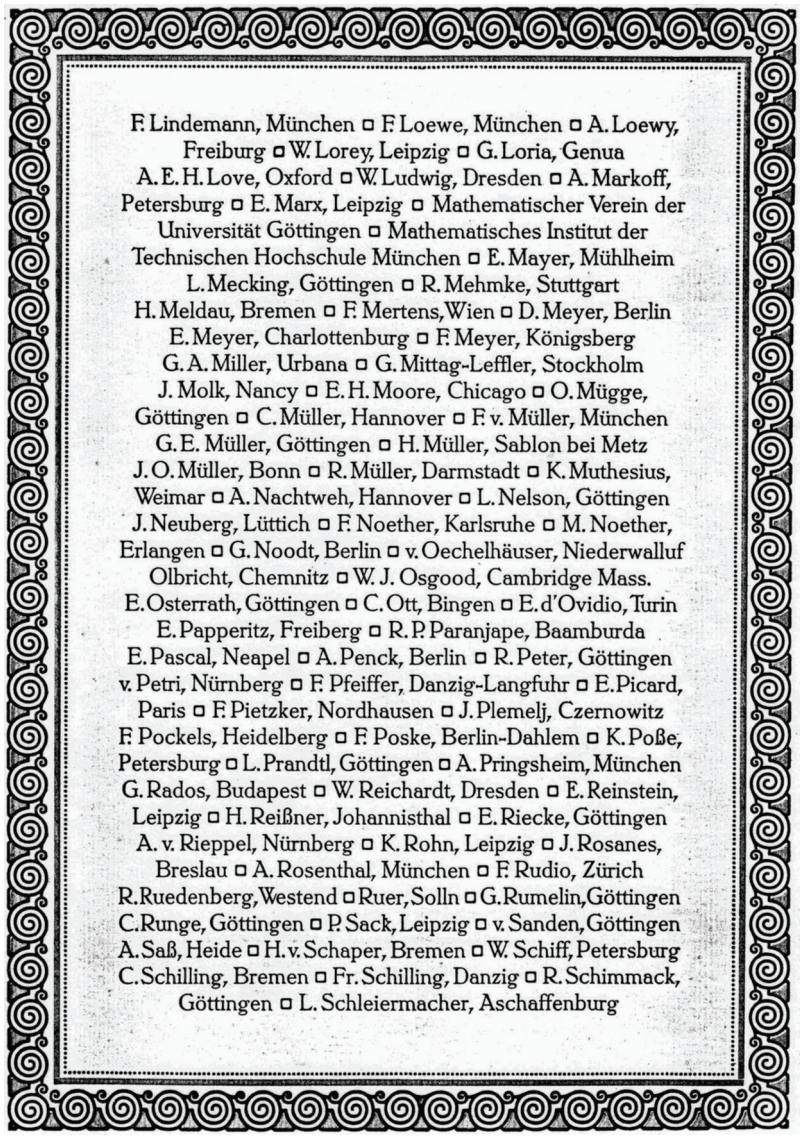
Tokio ◻ Ph. Furtwängler, Wien ◻ M. Gebhardt,
Blasewitz-Dresden ◻ N. Gernet, Petersburg ◻ Gessting, Kassel
K. Giebel, Zeitz ◻ W. Giesing, Merscheid ◻ M. Girndt,
Magdeburg ◻ T. Glauner, Wittenberg ◻ E. Götting, Göttingen
K. Goldzieher, Budapest ◻ P. Gordan, Erlangen ◻ J. P. Gram,
Kopenhagen ◻ W. Greve, Geestemünde ◻ E. Grimsehl,
Hamburg ◻ M. Grübler, Dresden ◻ G. B. Guccia, Palermo
A. Gutzmer, Halle ◻ J. Hadamard, Paris ◻ H. Hahn, Czernowitz
G. Hamel, Brünn ◻ F. Hanker, Geestemünde ◻ H. Happel,
Tübingen ◻ E. Hartmann, Frankfurt ◻ J. Hartmann, Göttingen
M. W. Haskell, Berkeley, Cal. ◻ R. Haussner, Jena ◻ Hecke,
Göttingen ◻ G. Heckmann, Berlin ◻ P. Heegaard, Kopenhagen
E. Hellinger, Marburg ◻ R. Helmert, Potsdam ◻ L. Henneberg,
Darmstadt ◻ O. Henrici, Chandler'sford ◻ K. Hensel, Marburg
G. Herglotz, Leipzig ◻ W. Heß, Bamberg ◻ K. Heun, Karlsruhe
K. Hiemenz, Darmstadt ◻ E. Hilb, Würzburg ◻ D. Hilbert,
Göttingen ◻ R. Hildebrand, Leipzig ◻ E. W. Hobson, Cambridge
H. Hochschild, Charlottenburg ◻ F. Höck, Perleberg
B. Hoffmann, Rawitsch ◻ A. Höfler, Wien ◻ O. Hölder, Leipzig
A. Hölscher, Duderstadt ◻ H. Hort, Essen ◻ A. Hurwitz, Zürich
Th. Indlien, Tomsk ◻ C. Jaccottet, Lausanne ◻ E. Jahnke,
Berlin ◻ C. Juel, Kopenhagen ◻ D. Katz, Göttingen
G. Kerschensteiner, München ◻ L. Kiepert, Hannover
O. Knoblauch, München ◻ A. Koch, Göttingen ◻ H. v. Koch,
Stockholm ◻ P. Koebe, Leipzig ◻ J. König, Budapest
K. Körner, Wolfenbüttel ◻ E. Köster, Flensburg ◻ A. Kötz,
Göttingen ◻ A. Krager, Karlsruhe ◻ A. N. Kriloff, Petersburg
L. Krüger, Potsdam ◻ Krupp v. Bohlen und Halbach, Hügel
Kuße, Danzig-Langfuhr ◻ E. Landau, Göttingen ◻ V. v. Lang,
Wien ◻ E. Lange, Dresden ◻ K. Laubert, Kassel ◻ M. Laue,
Zürich ◻ A. Lehmann, Königsberg ◻ F. Leo, Göttingen
B. Lepsius, Berlin ◻ W. Lexis, Göttingen ◻ H. Liebmann,
München ◻ W. Lietzmann, Barmen ◻ C. v. Linde, München

Fig. 1.2 N.[adjeschda] v. Gernet had completed the Women's Courses in St Petersburg, studied under Felix Klein and David Hilbert, obtained her doctorate under the latter and was finally able to obtain a university position in St Petersburg. – Gessting = Gustav Gersting teacher examination (Göttingen 1906), teacher at the girl's secondary school in Kassel. – Goldzieher = Karl Goldziher. – "Th. Indlien" (Tomsk) = Klein's student Theodor Molien from Russia. – A. Krager (Karlsruhe) = Adolf Krazer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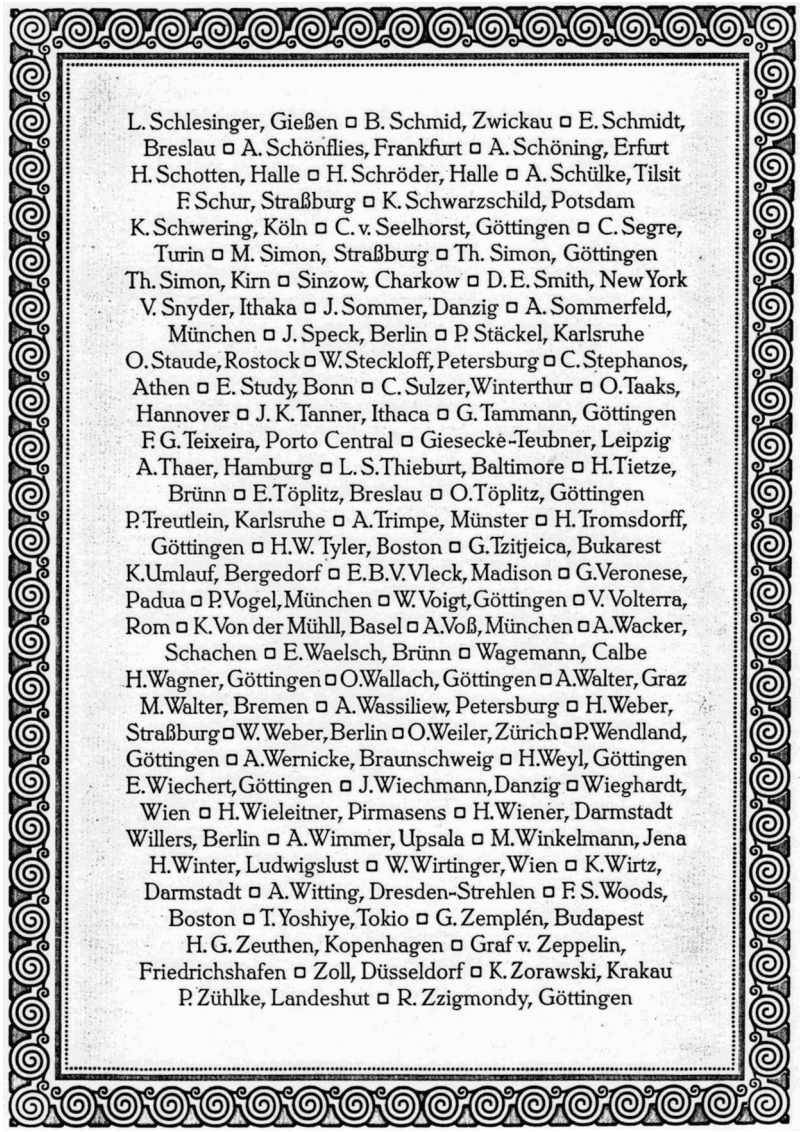
Freiburg ◻ W. Lorey, Leipzig ◻ G. Loria, Genua
A. E. H. Love, Oxford ◻ W. Ludwig, Dresden ◻ A. Markoff,
Petersburg ◻ E. Marx, Leipzig ◻ Mathematischer Verein der
Universität Göttingen ◻ Mathematisches Institut der
Technischen Hochschule München ◻ E. Mayer, Mühlheim
L. Mecking, Göttingen ◻ R. Mehmke, Stuttgart
H. Meldau, Bremen ◻ F. Mertens, Wien ◻ D. Meyer, Berlin
E. Meyer, Charlottenburg ◻ F. Meyer, Königsberg
G. A. Miller, Urbana ◻ G. Mittag-Leffler, Stockholm
J. Molk, Nancy ◻ E. H. Moore, Chicago ◻ O. Mügge,
Göttingen ◻ C. Müller, Hannover ◻ F. v. Müller, München
G. E. Müller, Göttingen ◻ H. Müller, Sablon bei Metz
J. O. Müller, Bonn ◻ R. Müller, Darmstadt ◻ K. Muthesius,
Weimar ◻ A. Nachtweh, Hannover ◻ L. Nelson, Göttingen
J. Neuberg, Lüttich ◻ F. Noether, Karlsruhe ◻ M. Noether,
Erlangen ◻ G. Noodt, Berlin ◻ v. Oechelhäuser, Niederwalluf
Olbricht, Chemnitz ◻ W. J. Osgood, Cambridge Mass.
E. Osterrath, Göttingen ◻ C. Ott, Bingen ◻ E. d'Ovidio, Turin
E. Papperitz, Freiberg ◻ R. P. Paranjape, Baamburda
E. Pascal, Neapel ◻ A. Penck, Berlin ◻ R. Peter, Göttingen
v. Petri, Nürnberg ◻ F. Pfeiffer, Danzig-Langfuhr ◻ E. Picard,
Paris ◻ F. Pietzker, Nordhausen ◻ J. Plemelj, Czernowitz
F. Pockels, Heidelberg ◻ F. Poske, Berlin-Dahlem ◻ K. Poße,
Petersburg ◻ L. Prandtl, Göttingen ◻ A. Pringsheim, München
G. Rados, Budapest ◻ W. Reichardt, Dresden ◻ E. Reinstein,
Leipzig ◻ H. Reißner, Johannisthal ◻ E. Riecke, Göttingen
A. v. Rieppel, Nürnberg ◻ K. Rohn, Leipzig ◻ J. Rosanes,
Breslau ◻ A. Rosenthal, München ◻ F. Rudio, Zürich
R. Ruedenberg, Westend ◻ Ruer, Solln ◻ G. Rumelin, Göttingen
C. Runge, Göttingen ◻ P. Sack, Leipzig ◻ v. Sanden, Göttingen
A. Saß, Heide ◻ H. v. Schaper, Bremen ◻ W. Schiff, Petersburg
C. Schilling, Bremen ◻ Fr. Schilling, Danzig ◻ R. Schimmack,
Göttingen ◻ L. Schleiermacher, Aschaffenburg

Fig. 1.3 R.P. Paranjape = R.P. Paranjpye, Indian math. – R. Peter = Albert Peter, Prof. of Botany. – W. Schiff = Vera I. Schiff, (female) Prof. of Math. at the Women's Courses in St Petersburg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F. Schur, Straßburg ◻ K. Schwarzschild, Potsdam
K. Schwering, Köln ◻ C. v. Seelhorst, Göttingen ◻ C. Segre,
Turin ◻ M. Simon, Straßburg ◻ Th. Simon, Göttingen
Th. Simon, Kirn ◻ Sinzow, Charkow ◻ D. E. Smith, New York
V. Snyder, Ithaka ◻ J. Sommer, Danzig ◻ A. Sommerfeld,
München ◻ J. Speck, Berlin ◻ P. Stäckel, Karlsruhe
O. Staude, Rostock ◻ W. Steckloff, Petersburg ◻ C. Stephanos,
Athen ◻ E. Study, Bonn ◻ C. Sulzer, Winterthur ◻ O. Taaks,
Hannover ◻ J. K. Tanner, Ithaca ◻ G. Tammann, Göttingen
F. G. Teixeira, Porto Central ◻ Giesecke-Teubner, Leipzig
A. Thaer, Hamburg ◻ L. S. Thieburt, Baltimore ◻ H. Tietze,
Brünn ◻ E. Töplitz, Breslau ◻ O. Töplitz, Göttingen
P. Treutlein, Karlsruhe ◻ A. Trimpe, Münster ◻ H. Tromsdorff,
Göttingen ◻ H. W. Tyler, Boston ◻ G. Tzitjeica, Bukarest
K. Umlauf, Bergedorf ◻ E. B. V. Vleck, Madison ◻ G. Veronese,
Padua ◻ P. Vogel, München ◻ W. Voigt, Göttingen ◻ V. Volterra,
Rom ◻ K. Von der Mühll, Basel ◻ A. Voß, München ◻ A. Wacker,
Schachen ◻ E. Waelsch, Brünn ◻ Wagemann, Calbe
H. Wagner, Göttingen ◻ O. Wallach, Göttingen ◻ A. Walter, Graz
M. Walter, Bremen ◻ A. Wassiliew, Petersburg ◻ H. Weber,
Straßburg ◻ W. Weber, Berlin ◻ O. Weiler, Zürich ◻ P. Wendland,
Göttingen ◻ A. Wernicke, Braunschweig ◻ H. Weyl, Göttingen
E. Wiechert, Göttingen ◻ J. Wiechmann, Danzig ◻ Wieghardt,
Wien ◻ H. Wieleitner, Pirmasens ◻ H. Wiener, Darmstadt
Willers, Berlin ◻ A. Wimmer, Upsala ◻ M. Winkelmann, Jena
H. Winter, Ludwigslust ◻ W. Wirtinger, Wien ◻ K. Wirtz,
Darmstadt ◻ A. Witting, Dresden-Strehlen ◻ F. S. Woods,
Boston ◻ T. Yoshiye, Tokio ◻ G. Zemplén, Budapest
H. G. Zeuthen, Kopenhagen ◻ Graf v. Zeppelin,
Friedrichshafen ◻ Zoll, Düsseldorf ◻ K. Zorawski, Krakau
P. Zühlke, Landeshut ◻ R. Zzigmondy, Göttingen

Fig. 1.4 J.K. Tanner = John Henry Tanner, Cornell University – A. Trimpe = Aloys Timpe, PhD with Klein in 1905. – O. Weiler = Adolf Weiler, PhD with Klein in 1873, Swiss math. – "A. Wimmer, Upsala" = Anders Wiman, Uppsala

Darboux:

> Here in Erlangen I was at first very isolated, especially scientifically; my only interactions were with beginners. Since then, two older people have come here from Göttingen who work independently on geometry. I hope very much that this number increases next semester and that, in Erlangen, I will gradually succeed in establishing a school of geometric productivity such as that which had gotten off to such a great start in Göttingen while Clebsch was still there.[1]

As a *Privatdozent* (after his habilitation in January 1871) at the University of Göttingen, Klein worked closely with Alfred Clebsch and his students. Clebsch established a new form of collaboration, which Klein quickly adopted. We agree with this statement: "Clebsch belonged to a new type of mathematician: he was characterised by a lively pedagogical activity, a lively community with younger mathematicians, in a word: the founding of a school. [...] A circle of enthusiastic co-workers and students gathered around Clebsch: Gordan, Brill, Lüroth, Zeuthen, and the most famous: [Max] Noether and Klein."[2]

Klein's working style – orientated towards Clebsch – corresponded to his own inner urge to influence others and help them to make progress. This is well illustrated by a statement in a letter from Klein to the Norwegian mathematician Sophus Lie, his collaborator in the early 1870s: "Until now, there has been no mathematics in Bavaria, and the southern German student seems to be a lazy thinker. That said, my rather strong social drive – if this is understood as the desire to affect other people – will presumably be satisfied."[3]

As a *Privatdozent*, Klein had already supervised his first doctoral student, the German Joseph Diekmann, and sent his dissertation to Lie with the remark: "I helped the author a great deal. [...]."[4] In his last semester there, Klein also guided his first *foreign* doctoral student, the Swiss Adolf Weiler, in the direction of his research. Weiler had previously studied with Wilhelm Fiedler in Zurich, who became famous for the German editions of the textbooks on analytic, projective and algebraic geometry by the Irishman George Salmon; Fiedler incorporated new findings by Clebsch, Klein and others directly into his expanded editions. Klein wrote to Fiedler in July 1872: "This summer I am giving a two-hour lecture course on line geometry with the express intention of encouraging others to work in this direction. Mr Weiler, to whom you probably suggested to come here, is in the audience. He has outstanding geometric talent [...]."[5]

Due to Klein's subsequent surprise appointment to Erlangen in October of 1872 and Clebsch's early death (7 November 1872), Weiler completed his doctorate under

[1] [PARIS] 64: Klein to Darboux, November 29, 1872, emphasis original.

[2] Shafarevich (1983, 139–140) – See in detail Tobies (2021, 47–53).

[3] [OSLO] A letter from Klein to Lie dated August 3, 1872. – For the discrepancies that later emerged between Klein and Lie, see the recent analysis by Kay (2023).

[4] [OSLO] A letter from Klein to Lie dated July 29, 1871. – Cf. on this Tobies (2021, 107).

[5] Confalonieri et al. (2019, 92–93) (Klein's letter to W. Fiedler, dated July 20, 1872).

Klein at the University of Erlangen in 1873. Overall, Klein's efforts to create a productive geometric school in Erlangen were successful despite the small number of students. In the course of just five semesters there, he inspired not only the *Habilitation* of Aurel Voss, but also the doctoral theses of six mathematicians, four of whom went on to university careers, including Adolf Weiler, who eventually became a professor in Zurich.

It was typical of Klein to publicise the good results of his students in letters to colleagues and also in the journal *Mathematische Annalen* (*Math.Ann.* for short), for whose edition Klein (with Adolph Mayer) took over the main responsibility from 1876 onwards. For example, Klein informed Lie: "Two dissertations will soon be completed here: the first [=Weiler's] concerns my classification of second-degree complexes [. . .]."[6] Ten years later, when the Italians Corrado Segre and Gino Loria further developed the classification system to all possible cases of harmonic complexes, even discovering errors in Weiler's thesis, Klein decided to disseminate the results. He published them in "his" journal,[7] and remained on very good terms with these (and other) Italian mathematicians throughout his life.[8] Both, like Weiler, were among the donors of Liebermann's Klein portrait (see Figs. 1.3 and 1.4).

The aforementioned Aurel Voss attested the young Klein "a remarkable ability to identify in the works of others, the point that related specifically to his own ideas" and "a talent of directing each of his students to the topic that best suited the latter's particular gifts and stage of development."[9] Klein continued to support (and challenge) young mathematicians; he developed a method of intensive mentoring, good preparation of the seminar lectures, and regular discussions with each individual. The Austrian mathematician Wilhelm Wirtinger (see Fig. 1.4) reported on his semester in Göttingen:

> In the summer term of 1889 I went to Göttingen to Felix Klein. He read about abelian functions and partial differential equations in physics and involved the advanced students. We were Burkhardt, Haskell, Osgood, White and I in the seminar at the time. With Klein, I immediately found the richest encouragement and inspiration. He was extremely adept at specifying and processing the burgeoning ideas and devoted a great deal of his time and energy to periodic meetings with each individual.[10]
>
> Klein never hesitated to communicate his results, and he also openly explained the paths he took. [. . .] He introduced his students not only to the subject matter, but also to the art of

[6] [Oslo] A letter from Klein to Lie dated June 28, 1873. – Weiler, A. (1874). "Ueber die verschiedenen Gattungen der Complexe zweiten Grades." *Math.Ann.* 7, pp. 145–207.

[7] See Rowe (2016), Tobies (2021, 42).

[8] As professors in Turin, Segre and his former student Gino Fano had an impressive collection of Klein's writings, books and elaborated lecture courses in their personal library. See Chap. 5 by Erika Luciano in this book.

[9] Voss (1919, 286).

[10] [AdW Wien] Wirtinger 1939: 5. – On Wirtinger's relationship to Klein, see Peter Ullrich's contribution, Chap. 8.

> disposition and to scientific expression in general. Klein has allowed us to experience the manifold tensions and their solutions in mathematical problems.[11]

Felix Klein, who travelled to Paris, Italy, Great Britain and the United States several times and was friends with many foreign colleagues, had a reputation for guiding young mathematicians to their own creativity. Sophus Lie, who knew early on of Klein's desire to guide, work with and help young talents, wrote to his Norwegian student Elling Holst in 1874 recommending that he go to Klein: "Believe me, there is nothing for you in *Berlin* unless you were to be as lucky as I and meet another Klein there."[12]

At the time, Berlin was dominated by the mathematicians Ernst Eduard Kummer, Karl Weierstraß and Leopold Kronecker.[13] They had very different ideas about working with young people. Kummer is said to have likened the request for a dissertation topic to "a young man asking him to suggest a pretty young girl with whom he could fall in love".[14] In the last year of his life, Kronecker wrote in a letter to Georg Cantor, the creator of set theory and initiator of the *Deutsche Mathematiker-Vereinigung* (DMV, German Mathematical Society, founded in 1890):

> We neither want nor need a school, because in our absolutely clear science every new discovery can render the previous school knowledge worthless. [...] We can therefore expect nothing beneficial from such joint work. On the contrary, such work can only hinder the progress of mathematics.[15]

Internationally, however, there was talk of Klein's mathematical school. More and more foreign colleagues were sending their own students to study with him. For example, the Russian mathematicians Andrey A. Markov [Markoff] (see Fig. 1.3) and Nikolay Y. Sonin spoke of Felix Klein's mathematical school in their proposal to elect him as a corresponding member of the St Petersburg Academy.[16] On 26 December 1895, Charles Hermite congratulated Klein on his election to the St Petersburg Academy of Sciences and informed him (via his translator Léonce Laugel) that the Paris Academy intended to do the same at the earliest opportunity. Klein was elected on 17 May 1897, taking the place of the late

[11] Wirtinger (1919, 287–288). Original quotation: "Klein hat niemals mit der Mitteilung seiner Ergebnisse gekargt und auch die Wege auf denen er vordrang, hat er immer offen dargelegt. [...] Er hat seine Schüler nicht bloß in den Gegenstand, sondern auch in die Kunst der Disposition und die wissenschaftliche Ausdrucksweise überhaupt eingeführt."

[12] Quoted from Stubhaug (2002, 236).

[13] See in more detail, also on the Weierstraß school of function theory, Tobies (2023, pp. 1–5).

[14] Runge (1949, 40).

[15] Kronecker, Leopold (1892). "Auszug aus einem Briefe von L. Kronecker an Herrn Prof. G. Cantor, vom 18. September 1891." *Jahresbericht der Deutschen-Mathematiker-Vereinigung* [DMV] 1, pp. 23–25. See also Tobies (2023, pp. 2–3).

[16] [Archiv St. Petersburg] Fond 2, 810, Bl. 16–17 (Elected in December 1895). – See also Tobies (2018).

James J. Sylvester.[17] The Hungarian Julius [Gyula] König informed Klein that the Magyar Tudományos Akadémia in Budapest had elected him a foreign member in accordance with his application.[18] During his lifetime, Klein became a member of fifty-one academies and societies. In some cases, these memberships were initiated by younger mathematicians who had studied under him, such as the aforementioned Wilhelm Wirtinger in Vienna or Ernesto Pascal in Naples.[19]

The list of donors for the Felix Klein portrait painted by Max Liebermann (Fig. 1.1 to 1.4) could be a kind of signpost for Klein's extensive network. There is also evidence of links as far afield as India and Australia. This list has already been included in the appendix of Tobies (2021). In the meantime, however, some misspelled names in this printed greeting address have been identified, which has led to further discoveries. We are including the list here, with the errors corrected.

1.2 Felix Klein's Foreign Students

Early in his career Klein made a name for himself by demonstrating the non-contradictory nature of non-Euclidean geometry, and by using group theory to systematise the various geometric directions (his *Erlangen Programme* of 1872). His attempt to combine methods from different mathematical disciplines (geometry, algebra, function theory, number theory) became a special feature of his research and at the same time enabled him to supervise doctoral students in several disciplines, including applied fields.

Many of his works have been translated into English, French, Italian and other languages. The influence of his *Erlangen Programme*, which has also been translated into Polish, Russian and Hungarian since the 1890s, was and still is far-reaching. It was also distributed in Spain and Argentina (see the contribution by A. Ramirez) and translated into Japanese still in 1970 (see H. Kümmerle).[20] The Greek mathematician Constantin Carathéodory (see Fig. 1.1), who studied under Klein and was to become his successor in Göttingen in 1913, wrote enthusiastically about the scope of Klein's *Erlangen Programme* as a classification principle for mechanics and general relativity.[21] Eilenberg and Mac Lane saw their category theory in the footsteps of the programme: "This may be regarded as a continuation

[17] "Nous sommes heureux d'avoir à vous annoncer que, dans la Séance de se jour, l'Académie vous a nommé à la place de Correspondant, devenue Vacante dans la Section de Géométrie, par suite du décès de M. Sylvester." ([UBG] Cod. MS. F. Klein 114: 25). – See in more detail in Tobies (2016).

[18] [UBG] Cod. MS. F. Klein 10: 531 (G. König's letter to Klein, dated 8 May 1899).

[19] Pascal (see Fig. 1.3) attended Klein's cours on Abelian functions in the winter semester 1888/89 and in the summer semester 1889, while Wirtinger only enrolled for one semester. [UBG] Cod. MS. F. Klein 7 E, pp. 161v, 162v). – For Pascal see Maria Giulia Lugaresi's section in this book.

[20] These results are missing in Rowe (2025), who presents a new translation of the *Erlangen programme* into English, combined with a selected insight into the history of its reception.

[21] Carathéodory (1919, 300) – In this book Christine Phili explores Klein's relations with Greek mathematicians, including C. Stéphanos' early attempt to publish Klein's programme in French translation.

of the Klein *Erlanger Programm*, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings."[22] It has also influenced philosophical systems. Schiemer (2020) described Klein's programme as an important origin of structuralism in the philosophy of mathematics. Ernst Cassirer, see Ihmig (1998); and Alfred North Whitehead are particularly worth mentioning.[23]

Let us stick to the historical order of some of the links with younger foreign students who studied with Klein. Even before the mentioned Elling Holst the Swede Victor Bäcklund, who already held a doctorate, came to Erlangen in 1874 on a six-month travelling scholarship. Klein published Bäcklund's work, and he managed to get Holst a Norwegian travelling scholarship to continue his studies with him at the Polytechnic in Munich.[24] Holst taught and researched mainly in the field of geometry and eventually got a position (docent) at the University of Oslo in 1894. Later, during Klein's time in Göttingen, other Scandinavians studied with Klein, for example the Norwegian Alf Guldberg in 1888–1889, the Dane Poul Heegaard in 1894, and Elizabeth Stephansen in 1902/03, the first Norwegian woman to obtain a doctorate in mathematics.[25]

Klein's basic courses in Munich were attended by many (engineering) students, including numerous from abroad, especially from Austria-Hungary. Apart from Holst, only four other foreigners, the German-speaking Bohemian Anton Puchta and the Czech Ludvík Kraus (see M. Bečvářová's section) and two Italians, attended Klein's advanced courses in Munich.

When Klein moved to the *University of Leipzig* in October 1880, he was joined for the first time by a Frenchman, Georges Brunel, a Briton, Arthur Buchheim,[26] and mathematicians from the USA, Russia, and other European countries. An overview of their involvement in Klein's courses and his inspiration can be found in Tobies (2021). Some aspects are explored in more detail in this book.

[22] Eilenberg and Mac Lane (1945, 237).

[23] Thanks to Michael Rahnfeld for analysing the Whitehead literature, see esp. Kraus (2018, 145).

[24] See Tobies (2021). Holst, E. (1877). "Ein paar allgemeine metrische Sätze für algebraische Curven." *Math.Ann.* 11, pp. 341–46. – For his further publications see Poggendorff (1898–1926), vol. III, p. 652; vol. IV.1, p. 660.

[25] Elizabeth Stephansen studied on a scholarship at Zurich Polytechnic (with Klein's student Hurwitz). After completing her studies, she got a job as a teacher in Norway and worked on a dissertation, with which she was able to complete her doctorate in absentia under Heinrich Burkhardt at the University of Zurich in 1902. Burkhardt had previously habilitated with Klein in Göttingen. See also Siegmund-Schultze (2024), and Tobies (2020), which lists all the women who attended Klein's courses.

[26] June Barrow-Green and Brigitte Stenhouse will analyse Klein's influence in Britain in this book.

Brunel's studies with Klein are of particular importance in connection with the correspondence between Felix Klein and Henri Poincaré[27] It should be especially emphasized that Darboux had recommended Brunel to Klein, who led him to his first publication, and that Brunel contacted Poincaré on Klein's behalf. Brunel was to help Poincaré understand Klein's results. However, in his letters to Poincaré, whom he did not know personally, Brunel made strong French nationalist statements.[28] This may have influenced Poincaré to be reluctant to share his own concrete methods with Klein. The collaboration Klein wanted (as he had them with many other mathematicians) turned into a kind of competition.

Parshall and Rowe (1994) provides an excellent study of Klein's impact in the United States. Nevertheless, there are still new aspects that have been discovered. For example, the list (Figs. 1.1 to 1.4) contains 14 US-Americans, including some not mentioned by Parshall & Rowe, such as Arthur Bowes Frizell and Peter Field, both of whom studied in Göttingen and completed their doctorates in the USA.[29] In Chap. 6 of this book, H. Heller gained new insights into the life and work of the American M.W. Haskell, who had already spent a semester with Klein in Leipzig (without enrolling in a course) before moving to the *University of Göttingen* with his teacher to complete his doctorate.

In Göttingen, Klein also established probability theory, actuarial mathematics and statistics as fields of teaching and research. In this book (Chap. 15), H. Fischer analyses the seminars Klein held to promote these fields.

We should also mention another dimension of Klein's "influence", namely the promotion of a collegial and scientific culture at foreign universities, modeled on what Klein's former students had previously experienced in Göttingen. Klein's model was important in various places in the world, e.g. in Italy (Turin, Naples

[27] See the most recent annotated edition (Nabonnand et al. 2024). Poincaré died on 17 July 1912 and is therefore not contained in the list of donors, but he belonged to the seventy-one people, who signed an appeal for donations for funds for the portrait in early 1912, see Tobies (2021, 613).

[28] See the analysis of the Darboux – Klein correspondence (Tobies 2016; Tobies 2021, 244–245), and the letters written by Brunel to Poincaré. "il ne faut pas croire pour cela que je me plaise beaucoup au milieu de ce peuple; plus j'apprends à le connaître et plus je le déteste." […] "Je dois dire cependant que relativement j'ai toujours trouvé que Mr Klein était le plus aimable et le plus obligeant. Enfin, quels qu'ils soient, il faut savoir profiter de ce qu'ils ont de bon; c'est toujours là ce que je me suis dit et c'est même pour cela que je suis en Allemagne. […] Je me mets tout entier à votre service. Français, notre devoir est de combattre les Allemands par tous les moyens possibles, mais loyalement." Brunel's letter to Poincaré, dated June 22, 1881. Poincaré, H., "La correspondance d'Henri Poincaré avec des mathématiciens de A à H." (1986), pp. 91–92. – Klein later managed to get Brunel to write an article for the Encyklopädie, "Bestimmte Integrale" [Definite integrals] (1899), in vol. 2.1.1, pp. 135–188.

[29] Both are listed in the *math genealogy project*. Frizell joined the German Mathematical Society in 1906 (see Toepell 1991, 115); after receiving his doctorate from the University of Kansas in 1910, he became a professor at the Nautical School in Kansas. – Peter Field was one of Klein's 'scientific grandsons'; his doctoral thesis (*On the Form of Unicursal Quintic Curves*, Cornell University, 1902) was supervised by Klein's student Virgil Snyder. In the years 1908–09 Field studied in Göttingen, where Klein inspired him to further results, published in *Math.Ann.*, vol. 67 and 69, and in *Zeitschrift f. Mathe. & Physik*, vol. 61, see Poggendorff, vol. V, p. 364.

esp.), in Russia (Moscow, Tomsk), Japan (Tohoku/Sendai), Budapest, Prague, in the United States, in India (Calcutta), Spain and Argentina, and also at Polish universities, especially after Poland gaines its independence. In Smyrna (then Greek), Carathéodory orientated himself on Klein's university model, which also provided for the promotion of East European / oriental languages.[30]

1.2.1 The Italians: Ricci-Curbastro, Bianchi, Veronese, and Others

Felix Klein had already established very good contacts with a number of important mathematicians since his trips to Italy in 1874 and 1878.[31] After Klein had visited Enrico Betti, Ulisse Dini and others in Pisa in April 1878, Gregorio Ricci-Curbastro came to Munich in the autumn on a scholarship. Ricci attended Klein's courses on number theory (1878/1879) and algebraic equations (1879), and participated in the seminars. There he gave two lectures: on "Certaines équations du degré 2^n", and on a paper by Henry Stephen Smith, which had been published in French in an Italian journal: "Les courbes modulaires: Rapport sur un mémoire de M. Stephen Smith communiqué à la Académie Royale des Lyncées, 1877".[32] Klein analysed the latest literature in the seminars and gave several talks on his own results. This increased Ricci's knowledge of the results of Riemann, Lipschitz and others. P. Speziali wrote: "Ricci greatly admired Klein, and his esteem was soon reciprocated [...]".[33]

When Ricci later developed tensor analysis with his student Tullio Levi-Civita, Klein requested that these results be published in *Math.Ann.*[34] Georg Pick would later refer Albert Einstein to the article by Ricci and Levi-Civita, when they were both professors in Prague (Pick had studied with Klein for two semesters and continued to work closely with him).[35] Although Einstein still needed help to grasp its significance and understand it mathematically, the work of Ricci and Levi-Civita provided the crucial foundation for his general theory of relativity.[36] These results would form the basis of Klein's own work on this topic, to which he also

[30] For Italy Chap. 5 by E. Luciano and Chap. 7 by M.G. Lugaresi, for Japan H. Kümmerle (Chap. 12), for Prague M. Bečvářová (Chap. 3) , for Greece Ch. Phili (Chap. 9), A. Ramírez Ogando for Spain and Argentina (Chap. 14); E. Mühlhausen on W.H. Young's attempt at the University of Calcutta (India, Chap. 10). Nicolas Michel looked at the conditions at the University of Chicago (see Report 2023, 2733–2736).

[31] See Tobies (2021, 153–158).

[32] [PROTOKOLLE] vol. 1.2, pp. 75–78 (Lecture, dated 28 July 1879).

[33] Speziali, Pierre. "Ricci-Curbastro, Gregorio." Dictionary of Scientific Biography.

[34] Ricci-Curbastro, Gregorio; Levi-Civita, Tullio (1901). "Méthodes de Calcul différentiel absolu et leurs applications." *Math.Ann.* 54, pp. 125–201. – For Ricci see also Poggendorff, Vol. III, p. 1118.

[35] Tobies (2023, 34).

[36] Klein (1927, 189–195, 205).

introduced Emmy Noether.[37] In 1899 Ricci had become a member of the DMV, which published a speech by him in German in its 1902 annual report, in which he drew a historical arc from Euclid to non-Euclidean geometry.[38]

Luigi Bianchi arrived in the summer term of 1879 and stayed for three semesters. Unlike Ricci at the time, he had an excellent command of German and enjoyed an intensive collaboration with Klein, who first asked him to talk about his doctoral thesis, and he invited him to write an article about it for the *Math.Ann.*[39] Klein included Bianchi in his field of research and recommended topics for five further lectures. Thus, Bianchi made contributions to Klein's level theory [*Stufentheorie*] and spoke about the tetrahedral irrationality (on 29 May and 5 June 1880) and about the icosahedral irrationality (on 13 June, 4 and 11 July 1880).[40] In Klein's work "Über unendlich viele Normalformen des elliptischen Integrals erster Gattung" [On Infinitely Many Normal Forms of the Elliptic Integral of the First Kind], which was first presented on July 3, 1880 at the Academy of Sciences in Munich and was an effort to expand his level theory with the help of doubly periodic functions, Klein acknowledged Bianchi's results as follows: "At my request, Mr. Bianchi recently investigated the fifth level, and what I will communicate below are essentially results discovered by him."[41] Bianchi expanded the proofs in an additional article, of whose beginning he thanked Klein "for his many suggestions and support in my work."[42] About the relationship between Bianchi's work and his own, Klein later explained:

> By treating, in the summer of 1880 (*Math.Ann.* 17), the elliptic curves that I would later call elliptic normal curves of the 3rd and 5th order by means of the σ function, he overcame my reservations about using this resource and that of theta series. He [Bianchi] thus did his best to build a bridge from my work to the developments of Weierstraß's school, particularly to my friend [Ludwig] Kiepert's studies, which were written around the same time."[43]

Conversely, Bianchi wrote to Klein: "I will never forget what you have done for me. And if I can be useful to you, remember that I am entirely at your disposal."[44]

Bianchi soon obtained a professorship in Pisa, where Ricci became his colleague; both – it is well known – developed methods (*Bianchi identities* for the Riemann tensor), which play an important role in the general theory of relativity. Bianchi

[37] See in detail in particular (Tollmien 2023, 273–283).

[38] Ricci, Gregorio (1902). "Anfänge und Entwicklung der neueren Auffassungen der Grundlagen der Geometrie" (held in Padua on 5 November 1901). *Jahresbericht der DMV* 11, pp. 382–403.

[39] Bianchi, L. (1880). "Ueber die Flächen mit constanter negativer Krümmung." *Math.Ann.* 16 (1880), pp. 577–82.

[40] Bianchi's seven lectures are recorded in [PROTOKOLLE] vol. 1.2, pp. 94–97; and vol. 2, pp. 6–10, 12–27.

[41] Klein (1921–1923), Vol. III, pp. 179–185, at p. 183.

[42] Luigi Bianchi, "Ueber die Normalformen dritter und fünfter Stufe des elliptischen Integrals erster Gattung," *Math.Ann.* 17 (1880), pp. 234–62, quotation p. 234.

[43] Klein (1921–1923), Vol. III, p. 6 – Ludwig Kiepert, a student of Weierstraß and a friend of Klein.

[44] [UBG] Cod. MS. F. Klein 8: 91 (Bianchi's letter to Klein dated August 14, 1880).

is famous for his work on differential geometry, about which he also wrote a comprehensive textbook *Lezioni di geometria differenziale* (Pisa 1894), which went through two German editions (1899, 1910) and was emphasised by Felix Klein.[45] Later, in February of 1924, Klein successfully arranged for Bianchi to be made a corresponding member of the Göttingen Academy of Sciences.[46]

The next Italian to join Klein was Giuseppe Veronese. As the son of poor parents, Veronese studied at Zurich *Polytechnic* from 1872 to 1876, where he was influenced by the above-mentioned Wilhelm Fiedler. Veronese then became Luigi Cremona's assistant in Rome. From there, he wrote to Klein in Munich on 1 May 1880, enclosing two notes of his own "Ueber einige bemerkenswerte Gruppen von Punkten, Geraden u. Ebenen, Kegelschnitten u. Flächen 2^{ten} Grades" [On some remarkable groups of points, straight lines and planes, conic sections and surfaces of the second degree], which were related to Klein and Lie's work on W-curves.[47] Veronese wished to study with Klein, who invited him to Leipzig, where he just had received the first professorship at a German university dedicated solely to geometry. Klein encouraged Veronese in his geometric interests, even if he himself was now more broadly orientated – as he informed Fiedler:

> [...] I have become somewhat alienated from geometry itself. I may always have been a geometer at the bottom of my mathematical thinking. One has been explicitly entrusted with the representation of geometry in Leipzig, as you probably know. For this reason I do not think I can change my present affairs, which, out of a certain inner necessity, are gradually embracing wider areas of abstract mathematics. But I will always be forced to give lectures on purely geometrical topics.[48]

Klein first asked Veronese for an overview of his previous work on projective configurations, which he handwrote in the protocol book under the title „Ueber einige merkwürdige Configurationen" [On some unusual configurations].[49] In the following summer semester of 1881, Veronese gave a presentation entitled "Ueber die Darstellende Geometrie im Raume von 4 Dimensionen" [On descriptive geometry in the space of 4 dimensions] (25 April 1881).[50] Klein, who quickly thought his way into other areas, wrote to Fiedler: "You may know that G. Veronese has been here since the beginning of winter. I am currently busy discussing a paper with him that he has delivered on the synthetic geometry of n-dimensional

[45] See Klein (1927, 148) – Bianchi also refers to Klein's results in his textbook.

[46] See Tobies (2021, 585).

[47] [UBG] Cod. MS. F. Klein 12: 42 (Veronese's letter to Klein, dated May 1, 1880). – For the works on W-curves see Tobies (2021, 78–80).

[48] Confalonieri et al. (2019, 111) (Klein's letter to W. Fiedler, dated July 30, 1880).

[49] [PROTOKOLLE] vol. 2, pp. 80–92 (Veronese's lecture on 3 January 1881).

[50] [PROTOKOLLE] vol. 3, pp. 1–13 (Lecture by Veronese on 25 April 1881 – Felix Klein's 32nd birthday).

space."[51] This work was published in the *Math.Ann.*, in which Veronese expressly thanked "Prof. Klein for his many suggestions and support during my mathematical studies in Leipzig".[52] After his studies in Leipzig, Veronese was appointed professor of descriptive geometry in Padua. He became known for his non-Archimedean geometry (1890), the consistency of which was doubted by Giuseppe Peano but recognised by David Hilbert in his *Grundlagen der Geometrie* (1899). Klein also appreciated Veronese's theory, but raised the question of whether it was appropriate [*zweckmäßig*] to deal with it.[53]

Two other Italian mathematicians studied with Klein in Leipzig; Francesco Gerbaldi came for the summer semester of 1883, and Giacinto Morera for two semesters from autumn 1883. They took part in Klein's research seminars, which at the time were dedicated to the theory of elliptic functions. Klein promoted publications by both of them.[54] Gerbaldi became a member of the German Mathematical Society (DMV) in 1897.

Klein's relations with Italian mathematicians, especially with representatives of algebraic geometry in Turin, were close and are generally well analysed.[55] Erika Luciano and C.S. Roero stressed, that Klein, in particular, was a "reference interlocutor" for Segre, Gino Fano, Federigo Enriques and Guido Castelnuovo.[56] Klein exerted a notable influence on mathematical studies, on the teaching of mathematics, on publishing activity, and on the organization of cultural and academic life in Turin. It was there that he received his first honorary doctorate in 1880 (out of a total of ten). Klein recruited the aforementioned Italians as authors for his major project, the *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* (Leipzig: B.G. Teubner, 1898–1935), Encyklopädie for short; a total of 16 contributions were written by eleven Italian scientists, most of them for Vol. 3 (Geometry). Ten Italians honoured Klein by supporting the portrait in 1912 (Fig. 1.1, 1.3, and 1.4).

In Göttingen, Alfonso Sella, later professor of physics in Rome, attended Klein's lectures on potential theory (in 1888) and partial differential equations in physics

[51] Confalonieri et al. (2019, 120) (Klein's letter to W. Fiedler, dated June 10, 1881).

[52] Veronese, G. (1881). "Die Anzahl der unabhängigen Gleichungen, die zwischen den allgemeinen Charakteren einer Curve im Raume von *n* Dimensionen stattfinden." *Math.Ann.* 18, p. 448; and Veronese, G. (1882). "Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens." *Math.Ann.* 19, pp. 161–234; acknowledgement, p. 162.

[53] Klein (1928, 156–157) English ed. (2016, p. 170). – Veronese's non-Archimedean geometry was included in his book *Grundzüge der Geometrie von mehreren Dimensionen und mehreren Arten gradliniger Einheiten in elementarer Form entwickelt* (Leipzig: B.G. Teubner, 1894; Italian Original: Padova 1891).

[54] The main aspects of their relationship with Klein are presented in Tobies (2021, 247–248) – Both otherwise only published in Italian journals, see Poggendorff, Vol. III, IV, and V.

[55] See in particular (Coen 2012; Casnati et al. 2016), and Erika Luciano's section in this book.

[56] Luciano, E.; Roero, C.S. (2016). "Corrado Segre and His Disciples: the construction of an International Identidy for the Italian School of Algebraic Geometry," in Casnati et al., 93–241.

(1888/1889 and summer 1889), as well as lectures by the theoretical physicist Woldemar Voigt. The above-mentioned Pascal was enrolled for the Abel functions for two terms (see M.G. Lugaresi's section). Gino Fano, who had translated Klein's *Erlangen Programme* (inspired by Segre), arrived in autumn 1893 for two semesters.[57] His seminar talk shows Klein's further turn to approximation theory and applications in physics.[58] Fano became even a serious candidate for Klein, whom he proposed for a professorship in Göttingen.[59]

1.2.2 Hungarian Mathematicians with Klein: Rados and Beke

The five Hungarian mathematicians listed in Fig. 1.1, 1.2, 1.3, and 1.4 all held positions in Budapest. Among them is the above mentioned Julius [Gyula] König, who was the same age as Felix Klein. As early as Easter 1873, he and Moritz [Mór] Réthy had attended the mathematicians' meeting in Göttingen, which was initiated by Alfred Clebsch and organised by Klein.[60] J. König was introduced to mathematics by Leo Königsberger in Heidelberg. In 1874, König was appointed full professor at the Royal Hungarian Technical (Joseph's) University in Budapest, which is famous for its leading role in mathematics. This Technical University was later known as the "Göttingen of the [Austro-Hungarian] Monarchy". Réthy, who had also studied in Germany (Göttingen, Heidelberg), became the first Hungarian professor of theoretical physics (in Klausenburg). In 1886 he moved to Budapest as professor of geometry and theoretical physics at the same university as Julius König. Réthy and J. König represented important fields (elliptic functions, non-Euclidean geometry – Bolyai – , hydrodynamics etc.), which had many connections to Klein's work, and both joined the DMV in 1893.[61] No wonder that a number of their students and other younger Hungarians found their way to Leipzig and Göttingen. In the order of their study under Klein, these include: Gusztáv Rados (Raussnitz) in 1884/1585 and 1885; Emanuel [Manó] Beke in 1893/1894; Ludwig [Lajos] Steiner in 1895; Karl [Károly] Goldziher in 1900/1901 and 1901; Leopold (Lipót)

[57] Klein, F. (1890). "Considerazioni comparative intorno a ricerche geometriche recenti." *Annali di matematica pura ed applicata* 17, pp. 307–43.

[58] [PROTOKOLLE] vol. 12, pp. 33–38 (G. Fano: "Allgemeine Bemerkungen über Fourier'sche Reihen (Vgl. Klein. Phys. Diffgl. II)". Fano gave this lecture on 13 and 20 June 1894. – In 1898/99, Adolfo Viterbi spoke in Klein's seminar; he was to become a professor of theoretical geodesy at the University of Pavia.

[59] Tobies (2021, 447).

[60] See ibid, pp. 117–121.

[61] Klein felt responsible for the DMV from the outset, was president three times and also invited many foreign colleagues and former doctoral students to become members (see in more detail (Tobies 2021, 367–372)).

Fejér in 1902/1903; Friedrich [Frigyes] Riesz in 1903/1904;[62] Győző Zemplén in 1904/1905; Alfréd Haar 1905/1906, 1906, 1907; 1909; Theodore von Kármán in 1906/1907, 1907/1908, 1908/1909; 1909; Dénes [Dionys] Kőnig – a son of Julius [Gyula] König – from 1904/1905 to 1907; Otto Szász and Aladár Visnya both in 1907/1908 and 1908.[63]

For the time being, we will only refer to Rados and Beke. The others worked mainly in Klein's interdisciplinary seminars (see point 3).

In the winter term of 1884/1885, Gusztáv Rados (Raussnitz until 1884/1885) attended Klein's courses without giving a lecture. In the summer of 1885, Klein treated the theory of algebraic functions of one variable in his seminar. He began by explaining that Riemann's theory can essentially be assumed to be known. His aim was to compare recent work with a geometric treatment of function theory with work based on an arithmetic approach.[64] With an already strong background in number theory from his studies with Julius König in Budapest, Rados spoke here on the work of Richard Dedekind and Heinrich Weber "Theorie der algebraischen Functionen einer Veränderlichen" [Theory of Algebraic Functions of One Variable][65] and on the content and methods of Kronecker's Festschrift, which was titled *Über den Zahlbegriff* [On the Concept of Number].[66] Rados analysed the latter book especially thoroughly, thereby demonstrating that his attitude toward research was closely aligned to Klein's own. He criticized Kronecker for limiting his focus to "the development of the characteristics [*Eigenschaften*] of rational whole numbers and functions without using methods from other areas of mathematics." Very much in the spirit of Klein, Rados promised, that he would use "all existing tools available to us." In his presentation, he outlined the fundamental concepts of Kronecker's work and praised the "algebraic materialisation [*Materialisirung*] of Kummer's ideal numbers or Dedekind's ideals," but also discussed other approaches to the topic.[67]

At the same time, Klein corresponded intensively with the aforementioned Georg Pick, whom he had persuaded to prepare Klein's work on elliptic module functions for the planned monograph.[68] There were still many unresolved issues on the topic.

[62] F. Riesz enrolled in the seminar in 1903/1904 ([UBG] Cod. MS. F. Klein 7 E, 219v); he did not give a lecture here, but he talked about his own new findings at the Göttingen Mathematical Society.

[63] See *Amtliche Verzeichnisse des Personals und der Studierenden der (Königlichen) Georg-August-Universität zu Göttingen*; [UBG] Cod. MS. F. Klein 7 E; and [PROTOKOLLE].

[64] [PROTOKOLLE] vol. 7, pp. 1–2 (Klein's introductory lecture, dated April 27 1885).

[65] Published in *Journal für die reine und angewandte Mathematik* 92 (1882), pp. 181–290.

[66] Kronecker's contribution to a *Festschrift* (*Philosophische Aufsätze*. Leipzig: Fues, 1887) for the philosopher Eduard Zeller must have been available to the seminar before publication. An extended version appeared in *Journal für die reine und angewandte Mathematik* 101 (1887), pp. 337–55.

[67] [PROTOKOLLE] vol. 7, pp. 27–50, 125–50 (lectures given on 1 June, 20 July, 27 July, and 3 August 3 1885). See also Tobies (2021, 249–250).

[68] The final product was Felix Klein and Robert Fricke: *Vorlesungen über die Elliptischen Modulfunctionen*. 2 vols. Teubner, Leipzig. Engl. trans. Arthur M. Dupre, *Lectures on the Theory of Elliptic Functions*. Beijing: Higher Education Press, 2017.

It is therefore not surprising that Klein inspired Rados and Pick to tackle a problem almost simultaneously.[69] Rados published his findings in Budapest in 1885.[70] From 1885 Rados held positions (as a full professor in 1893) at the same technical university as Julius König and Réthy. Klein and Rados remained in contact. From 1892 Rados was in charge of the Hungarian journal *Mathematikai és Physikai Lapok*. His first paper in the *Math.Ann.* appeared 1897.[71] When the Hungarian Academy of Sciences established the János Bolyai Prize (10,000 *Kronen*), Felix Klein and Gaston Darboux were the only two external members who were appointed to the prize committee in 1905, which also included Julius [Gulya] König and Gusztáv Rados.[72]

After his studies in Budapest, Emanuel [Manó] Beke became a mathematics teacher; he did his doctorate in 1884. He came to Göttingen on a one-semester scholarship in 1893/1894. Beke also joined the Göttingen Mathematical Society, which Klein had founded in 1892. In Klein's seminar on the theory of linear differential equations, Beke presented new results on homogeneous linear differential equations, which Klein immediately accepted for the *Math.Ann.*[73] In the same semester, women were first allowed to attend Klein's courses,[74] and Beke was enthusiastic about Klein's commitment to promoting women mathematicians and training secondary school teachers.[75] Back in Budapest, he was appointed professor at the College of Education (*Pädagogische Hochschule*) in 1895 and also taught at the first girls' secondary school there. In 1900, he became professor at the University of Budapest.

Beke and Klein later worked closely together as part of the International Commission for Mathematical Instruction (IMUK, today ICMI) founded in Rome in 1908. Klein was unable to travel to Rome, but was elected President of the IMUK in his absence. Beke said in his presentation in Rome: "By and large, we are all influenced by the movement that was inaugurated by Prof. Klein and further promoted by the German Commission."[76]

[69] See Tobies (2023, pp. 94–96), (Georg Pick's letters to Klein, dated 30 April, and 3 May 1885).

[70] Rados, G. (1885). "Az algebrai függvények elméletéhez" [On the Theory of algebraic functions]. *Matematikai és Természettudományi Értesíto* 3, pp. 185–87.

[71] Rados, G. (1897). "Theorie der adjungirten Substitutionen". *Math.Ann.* 48, pp. 417–24. See also Poggendorff, Vol. IV, p. 1206.

[72] See on this also Tobies (2016), and Tobies (2021, 250).

[73] [PROTOKOLLE] vol. 11, pp. 302–16 (Beke's lectures on January 10, 17 and 24, 1894). – Beke, E. (1894). "Die Irreducibilität der homogenen linearen Differentialgleichungen". *Math.Ann.* 45, pp. 278–94; and „Die symmetrischen Functionen bei den linearen homogenen Differentialgleichungen." Ibid., 295–300, elaborated in January and March 1894 in Göttingen.

[74] In 1893/94, the American Mary F. Winston and the Englishwoman Grace E. Chisholm, who completed their doctorates under Klein, began their work.

[75] [UBG] Cod. MS. F. Klein 8: 76A, 76B/1 (Beke's letters to Klein, dated March 8 and August 21, 1895).

[76] Quoted in Tobies (2021, 493) – For the history of the ICMI, see also (Furinghetti and Giacardi 2022).

1.2.3 Students and Elderly Mathematicians from Russia

Seven of the nine mathematicians from the Tsarist Russia[77] on the donators list (Figs. 1.1 to 1.4) are named with the location "Petersburg", including the two women. This is of course an expression of Klein's special relationships there, but it does not characterise the diversity of his contacts.

Klein's first student from Tsarist Russia, Theodor Molien, arrived in Leipzig in the autumn of 1883 after studying at the then German-speaking University of Dorpat (now Tartu, Estonia).[78] He stayed for three semesters, was twice a speaker at the seminars and was led to publishable results in Klein's research field at the time, the theory of elliptic functions.[79] With these results Molien was able to progress to the next academic level (Master's) at Dorpat, as he wrote to Klein.[80] Molien's most important paper "Ueber Systeme höherer komplexer Zahlen" [On systems of higher complex numbers] appeared in *Math.Ann.* [41 (1893), pp. 83–156], and earned him a doctorate from Dorpat. In 1900, he was appointed professor at the Technological Institute in Tomsk (from 1917 the Faculty of Physics and Mathematics of the University). Molien was the first professor of mathematics in Siberia and organised teaching, research and a library following the example of Klein.[81]

In addition to Molien, three older mathematicians from what was then Russia visited Klein in Leipzig, including Kiev-born Matvey A. Tichomandrickij and Kazan-born Alexander V. Vasilev [Wassiliew], both of whom had studied in St Petersburg and whose efforts were to contribute significantly to the dissemination of Klein's findings.

Tichomandrickij had already submitted a note (On the Converse Problem of Elliptic Integrals) for publication in *Math.Ann.* – before arriving in Leipzig. After giving a lecture in Klein's seminar (21 July 1884), a second note was published, which is still cited in modern scholarship.[82] In 1883, Tichomandrickij had just been appointed as a *Dozent* at the University Kharkov [Charkow] in 1883. He had a broad knowledge of Eastern European mathematical institutions. At Klein's request, Tichomandrickij sent him a multi-page list of researchers, institutions, main

[77] Until 1917, Tsarist Russia included Finland and parts of Poland, from where students also came. The Finn Alexander Leonard Hjelmman, for example, spoke at Klein's seminar in 1897/98 and became a professor in Helsinki. D. Ciesielska (Chap. 2) and J. Zwierzyńksa (Chap. 11) examine the links between Polish mathematicians and Klein in our book.

[78] See Siilivask (1985).

[79] Molien, Theodor (1885). "Ueber gewisse, in der Theorie der elliptischen Functionen auftretende Einheitswurzeln." *Berichte der Kgl. Sächsischen Gesellschaft der Wissenschaften zu Leipzig* 37, pp. 25–38 (submitted by Klein, 12 January 1885). – Ueber die lineare Transformation der elliptischen Functionen [On the linear transformation of elliptic functions] 24p., Dorpat 1885 (Master thesis).

[80] [UBG] Cod. MS. F. Klein 10: 1283 (Molien's letter to Klein, 14/26 December 1885).

[81] Molien's name is misspelled in Fig. 1.2, and I identified him only late. See Kanunow, N.F. (1983). *Fjodor Eduardowitsch Molin 1861–1941*. Moskwa: Isdatelstwo Nauka (Russ.).

[82] See Tobies (2021, 250–251), [PROTOKOLLE] vol. 6, 121–26; *Math.Ann.* 22 (1883), pp. 450–54; *Math.Ann.* 25 (1885), pp. 197–202.

research areas, Eastern European (Russian, Ukrainian, Polish, Hungarian, Czech) scholarly journals, and the researchers involved with them.[83] This list included three publications by the Academy of Sciences in St Petersburg, which published only in German and French, and Russian-language journals printed in Moscow, Kharkov, Kazan, Odessa, Kiev, Warsaw, and elsewhere.

On this basis, Klein decided to pursue exchanges between *Math.Ann.* and some of these journals, thereby attracting additional contributors and students. In 1891 Klein became a member of the Moscow Mathematical Society, initiated by Pavel A. Nekrasov [Nekrassoff], who published in the *Math.Ann.* the same year.[84] Klein was able to deepen his contact with Moscow when Boleslav K. Młodziejowski, born in Moscow, attended his lecture course (Analytical Mechanics) and seminar (Differential Equations) in 1891.[85] Młodziejowski, the son of a professor of pathology at the University of Moscow (who came from a Polish noble family) and a Czech mother, was to be appointed full professor of mathematics in Moscow in 1899. In 1891, he had thanked Klein in a letter "for everything I have gained through you in Göttingen" and informed him that he had carried out the "orders" [*Aufträge*] concerning Nekrasov and N.J. Zhukovsky exactly.[86]

A letter from Peter [Pjotr] M. Pokrowsky, who had ended his career in Moscow at the time and became a full professor at the University of Kiev in 1891, also shows that Klein intensified his contacts. Pokrowsky wrote: "My colleague and friend Professor Nekrassoff has forwarded your letter to the Mathematical Society in Moscow, from which it is clear that you are interested in new Russian works and would like to receive papers on them."[87] Pokrowsky, who joined the DMV in 1895, worked on the theory of hyperelliptic functions, which was also in Klein's area of interest at the time.

Vasilev, who visited Klein in Leipzig in the early 1880s, had studied in St Petersburg and Berlin. He also became a member of the DMV in 1893, lectured at its annual meetings, met Klein and initiated a Russian translation of Klein's *Erlangen Programme*, published in the *Bulletin de la Société de Kasan*, a work carried out by D.M. Sintsov [Sinzow], see also Fig. 1.4. Sintsov, whose research was based on that of Riemann, Klein and others, later visited Klein in Göttingen (1898). When Sintsov became president of the Mathematical Society of the University of Kharkov in 1906, Klein was accepted as a member. – In the 1890s, Vasilev invited Klein to join the "Lobachevsky Committee" in Kazan. Here Klein not only ensured that Sophus Lie

[83] [UBG] Cod. MS. F. Klein 12: 24 (Tichomandrickij to Klein, 14 November 1884).

[84] See Tobies (2021, 603–604).

[85] https://www.uni-math.gwdg.de/aufzeichnungen/klein-scans/klein/V10-1889-1892/V10-p186_low.jpg.

[86] [UBG] Cod. MS. F. Klein 10: 1269 (Młodziejowski to Klein, 28 October/9 November 1891). – Młodziejowski, mainly a geometer, later also published in the German journal, see for example Mlodziejowski, B. (1907). "Über aufeinander abwickelbare P-Flächen." *Math.Ann.* 63, pp. 62–84. See also Poggendorff, Vol. V, pp. 864–65.

[87] [UBG] Cod. MS. F. Klein 11: 370 (Pokrowsky's letter to Klein, on 2/14 December 1891).

received the first Lobachevsky Medal, but also that his own work on non-euclidean geometry was recognised. Klein was made an honorary member of the Imperial University of Kazan.[88] Vasilev had a good command of German and French and helped L. Laugel in Paris – the above-mentioned translator commissioned by Hermite – to translate Klein's papers, including Klein's third (longer) article on non-Euclidean geometry.[89] In the summer of 1906, one of Vasilev's students, Nikolaus N. Parfentiev [Parfentieff], came to Göttingen to attend Klein's course.[90]

The third person from Russia to attend Klein's courses in Leipzig was Ludwig von Struve. Born near St Petersburg into a famous family of astronomers, he followed in his father's footsteps. During a three-year stay abroad, in the winter term of 1884/85, he took the opportunity to attend a lecture course and a seminar by Felix Klein.

In the autumn of 1894, Vladimir P. Alekseyevsky [Wladimir P. Alexejewsky] and Theophil Friesendorff arrived in Göttingen, attended Klein's course on number theory and gave lectures in the seminar.[91] Alekseyevsky, born in the province of Novgorod and already a *Privatdozent* at the University of Charkow, had already published in German on the double gamma function he had discovered,[92] and was accepted into the Göttingen Mathematical Society. His later work included number theory, non-Euclidean geometry and the theory of automorphic functions. In 1904 he was appointed professor at the Technological Institute in Tomsk (as was Theodor Molien).[93] Friesendorff, who was born into a Polish family in the Minsk Governorate (then Russia), had completed his doctorate in St. Petersburg in 1893 and spent three semesters in Göttingen from 1894/1895 onwards (see Section 1.3.1).

[88] [UBG] Cod. MS. F. Klein 12: 200 (a letter from Vasilev to Klein dated 9 June 1895).

[89] L. Laugel & A. Vassilief [Vasilev] translated: Klein, F. (1897). "Sur la géométrie dite non euclidienne." *Annales de la Faculté des sciences de Toulouse: Mathématiques*, Sér. 1, 11 no. 4, pp.˜G1-G62; Klein, F. (1897). "Sur "l'arithmétization" des mathématiques." *Nouvelles annales de mathématiques* (journal des candidats aux écoles polytechnique et normale) Sér. 3, 16, pp. 114–28. – See also Tobies (2016).

[90] Parfentiev's activities had a number of links with Klein. He was involved in the organisation of the Lobachevsky competitions. In 1912 he was awarded the Lobachevsky Gold Medal. When he was appointed Head of the Department of Mechanics in 1930, he also gave extensive courses in elasticity theory and hydromechanics. https://ru.wikipedia.org/wiki/Власов,_Алексей_Константинович.

[91] On 31 October 1894, Friesendorff gave the first lecture of the semester on Dedekind's theory of irrational numbers ([PROTOKOLLE] vol. 12, pp. 87–88); Alekseyevsky spoke on 9 January 1895 on convergence of the series of functions (ibid., pp. 127–31).

[92] For Alekseyevsky's [Alexejewsky, Alexeevsky] work on the double gamma function, discovered in 1888, see Alexejewsky, W. (1894). "Ueber eine Classe von Functionen, die der Gammafunction analog sind". *Berichte der Sächsischen Gesellschaft der Wissenschften* 46, pp. 268–75.

[93] See also a recent paper by Yuri Neretin (2023): https://www.mathnet.ru/PresentFiles/40635/alex1.pdf, and the reception of Alekseyevsky's work in the article of Miklós Mikolás (1955). "Zur Theorie der Gammafunktion, der Zetafunktion und verwandter Funktionen." *Acta Mathematica Hungarica* 6: 381–438.

1.3 Klein's Interdisciplinary/Joint Research Seminars

Klein wanted and needed collaboration. His first joint seminar with Clebsch took place in the summer of 1872 ([PROTOKOLLE], vol. 1). At Erlangen, Klein included Paul Gordan in his seminar after he had secured the appointment of Gordan in 1874. In Munich, Klein succeeded in winning Alexander Brill for joint seminars for a time. In Leipzig, none of the older colleagues seemed suitable for Klein; but he was able to involve "his" *Privatdozent* Walther Dyck for a short time. In Göttingen, Klein had to wait a long time before his wish for collaboration was fulfilled. He had been appointed there in 1886 against the wishes of his mathematical colleagues H.A. Schwarz and Ernst Schering. At that time, the Göttingen Privatdozenten also rejected Klein's offer of joint seminars out of caution for their careers; Otto Hölder in particular was afraid of being caught between the chairs of Schwarz and Klein. Klein therefore had to wait for David Hilbert, whom he could regard as his scientific grandson.[94]

1.3.1 The First Joint Seminar in Göttingen

Klein managed to secure Hilbert's appointment on 1 April 1895 and announced a joint seminar with him for the summer term of 1895.[95] Klein also involved two of "his" *Privatdozenten*, Ernst Ritter and Arnold Sommerfeld, in the running of this seminar.

In 1895, almost all those present contributed papers, including the six women.[96] The lectures focused on approximation theory, methods of Fourier, Gauss, Chebyshev and his school, especially the methods of difference calculus (interpolation, sum and difference equations). Klein appreciated Chebyshev's approximation methods, emphasising as early as 1893 in Lecture 6 of his Evanston Colloquium "the highly interesting investigations of Tchebycheff [Chebyshev] on interpolation".[97] He went on to call Chebyshev as "an approximation mathematician par

[94] Hilbert had studied for a term with Klein in Leipzig, after completing his doctorate with Klein's student F. Lindemann in Königsberg, where he also benefited from Klein's student Adolf Hurwitz.

[95] Verzeichnis der Vorlesungen, Georg-Augusts-Universität Göttingen, SS 1895, p. 9. https://gdz.sub.uni-goettingen.de/id/PPN654655340_1895_SS.

[96] See Fig. 1.5 on the next page: Only L. Anhenn (Bremen, Germany) attended without giving a lecture. Among the foreigners who gave a lecture were: Ludwig [Lajos] Steiner (Ógyalla, Hungary), born into a Jewish family, had already obtained his doctorate in Budapest in 1893, only spent one semester in Göttingen, and later became a well-known geophysicist (in 1944, he felt compelled to commit suicide); John Henry Tanner, professor at Cornell University (see also Fig. 1.4), who recruited Klein's student Ernst Ritter for Cornell University (who died on the way there); Everett I. Yowell from Cincinnati, who became famous as an astronomer.

[97] Klein (1894, 230).

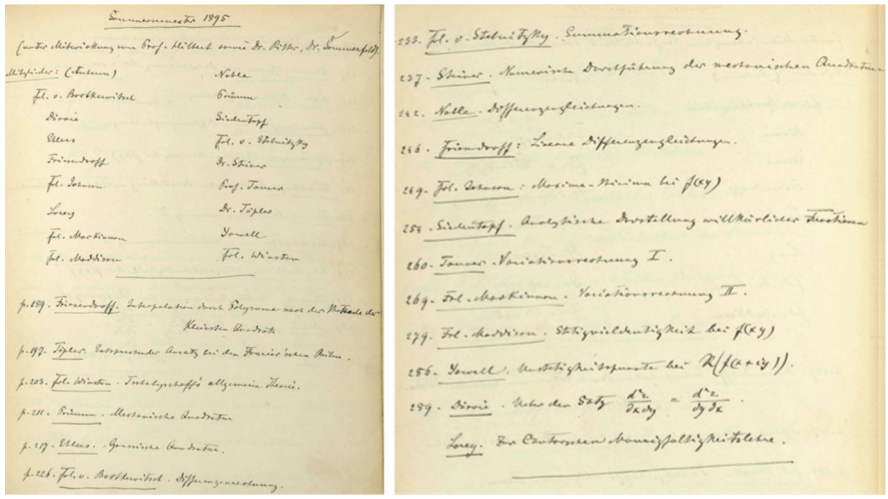

Fig. 1.5 Content of the seminar of summer semester 1895, which Felix Klein led with the assistance of D. Hilbert and the *Privatdozenten* E. Ritter and A. Sommerfeld (Klein's handwriting, [PROTOKOLLE], vol. 12, pp. 373–374)

excellence".[98] Hilbert followed the track (see Paul Kirchberger's thesis).[99] Steffens (2006, 131–134) defines Klein as a mediator in this area.

There were three people from St Petersburg in Klein's seminar who presented results about sum and difference equations, using a book by A.A. Markov in their lectures. They included the aforementioned Friesendorff and two women (see below). After Friesendorff had given two lectures, Klein asked Friesendorff to translate this book into German, and Markov gave Friesendorff permission to do so. He did this in 1895–1896, during which time he attended other lectures, including Klein's course on top theory. The book was published by Teubner as early as 1896 (Markoff 1896).[100] Friesendorff became a member of the DMV in the same year and was also elected as a member of the Mathematical Society in St Petersburg on 18 December 1896.[101] All in all, he was well prepared for his later career as a professor at the Institute of Electrical Engineering in St Petersburg.

[98] Klein (1928, 82) English ed. (2016, p. 87).

[99] Kirchberger, P. (1903). "Über Tschebychefsche Annäherungsmethoden". *Math.Ann.* 57: 509–40.

[100] https://archive.org/details/differenzenrech00markgoog/page/n6/mode/2up.

[101] http://www.mathsoc.spb.ru/trudy/Protocols/Protocols1890-99.pdf.

The women from St Petersburg, Helene von Bortkiewicz and Aleksandra Stebnicka [Alexandrine von Stebnitzky], who were born into Polish officer families,[102] were joined by women from Britain and America: Ada M. Johnson, Annie MacKinnon, Ada Isabel Maddison and the aforementioned Mary F. Winston. Klein officially reported that "six women have participated in our higher mathematics courses and have proven time and again that they are equal to their male fellow students in every way."[103] Mary F. Winston gave a lecture on Chebyshev's theory; not only did she do her doctorate with Klein, but she was also the first woman to publish an article in *Math.Ann.*[104] Of the seminar participants, the German Heinrich Dörrie and the American Charles A. Noble, defended their dissertations under Hilbert's supervision. Klein, who had supervised many Germans and foreigners for their doctorates, now increasingly entrusted this work to Hilbert, who was not to lead his first student to a doctorate until 1898.[105]

In the following semesters, Klein held further joint/interdisciplinary seminars with Hilbert and other newly appointed colleagues (see Table 1.1). Students usually attended courses given by Klein, Hilbert and other colleagues. The number of foreign students increased, including many from Hungary and Russia, which will be discussed in more detail in this section.

1.3.2 Further Women Mathematicians

Klein's aforementioned doctoral students Grace E. Chisholm (see E. Mühlhausen's section) and Mary F. Winston completed their doctorates in 1895 and 1897 respectively. If we follow the historical sequence, we find further women who came from St Petersburg. These women (as Bortkiewicz and Stebnicka) were graduates of the Higher Women's Courses (the famous Bestushev courses, see Chap. 11 by J. Zwierzyńska). Klein's remarkable contact with Wera Iosifovna v. Schiff (Ravitch) should be highlighted at this point (see also Fig. 1.3). V.I. v. Schiff was one of the first to attend the Bestushev courses in St Petersburg and, after a period of study in Paris, ended up teaching there as a professor and writing highly regarded textbooks.[106] She and her husband Peter [Pjotr] A. v. Schiff attended the

[102] Joanna Zwierzynska writes about their biographies and other St Petersburg educated women in this book.

[103] Felix Klein in Kirchhoff (1897, 241).

[104] Winston, Mary F. (1895). "Eine Bemerkung zur Theorie der hypergeometrischen Function." *Math.Ann.* 46, 159–60. – For her lecture on Chebyshev's theory she used the article "Sur quelques applications des fractions continues algebraiques" by Konstantin A. Posse [Poße] (see also Fig. 1.3), which Klein kept in a folder (entitled 'Russia 1887') in the students' working room [*Lesezimmer*], see [PROTOKOLLE] vol. 12, p. 210.

[105] For the doctoral theses under Klein's supervision see Klein (1921–1923), Vol. III, Appendix, pp. 11–13; for Hilbert's supervision see Hilbert (1935, 431–433).

[106] A biography can be found on the Russian Wikipedia page. There is no equivalent in any other language.

Table 1.1 Klein's seminars on pure and applied mathematics, which he led with colleagues

Semester	Additional seminar leaders	Title of the seminar announcements[a]	Presentations (foreign; female*)
1895	Hilbert, Ritter, Sommerfeld	Approximation analysis	17 (11;6)
1895/1896	Hilbert	Selected Chapters (Sel. Ch.) of Number Theory	9 (5;5)
1896	Hilbert	Sel. Ch. of Number Theory	8 (6;3)
1896/1897	Hilbert	Sel. Ch. of Function Theory	13 (3;1[a])
1897	Hilbert	Function Theory	12 (2;–[b])
1897/1898	Hilbert	Differential Equations of Mechanics	9 (3;1[c])
1898	Hilbert	Sel. Ch. of Mechanics (for advanced students only)	14 (2;–)
1898/1899	Hilbert	Sel. Ch. of Function Theory of real variables	14 (7;2[d,e])
1899	Hilbert	Sel. Ch. of Function Theory of real variables	12 (4;1[f])
1899/1900	only Klein	Sel. Ch. of Mechanics of deformable bodies[b]	20 (4;–)
1900	Abraham[c]	Sel. Ch. of elasticity theory[d]	11 (2;–)
1900/1901	Klein	Projective Geometry[e]	16 (–;–)
1901	[Sommer[f]]	Geodesy	11 (1; –)
1901/1902	only Klein	Sel. Ch. of Mechanics	13 (2;–)
1902	Schwarzschild[g]	Advanced mechanics with special reference to astronomical problems	15 (2;–)
1902/1903	Schwarzschild	Principles of mechanics[h]	16 (5;1[g])
1903	Schwarzschild	Building structure statics	15 (–;–)
1903/1904	Schwarzschild	Hydrodynamics	17 (1;–)
1904	Schwarzschild, Brendel[i]	Theory of Probability	18 (6;1[h])
1904/1905	Prandtl, Runge, Voigt, Simon[j]	Theory of Elasticity	18 (4;–)
1905	Runge, Simon, Prandtl	Electrical Engineering Technology	17 (1;–)
1905/1906	Hilbert,	Linear Differential	–[k]51
1906	Minkowski[l]	Equations and Auto-	13 (4;–)
1906/1907		morphic Functions	13 (6;–)
1907			–
1907/1908	Runge, Wiechert,[m] Prandtl	Hydro- and Aerodynamic Issues	11 (3;–)
1908	Runge, Wiechert, Prandtl	Hydro- and Aerodynamic Issues II[n]	10 (5;–)
1908/1909	Prandtl, Runge	Strength theory and statics of building structures	8 (1;–)
1909	Prandtl, Runge	Strength theory	9 (4;–)
1909/1910	Only Klein	On the Psychological Foundations of Mathematics	10 (2;–)

(continued)

Table 1.1 (continued)

Semester	Additional seminar leaders	Title of the seminar announcements	Presentations (foreign; female*)
1910	Klein on sick leave		
1910/1911	only Klein	Mathematical Instruction[o]	13 (1;3)
1911	Bernstein[p]	Actuarial Mathematics	10 (2;–)

Woman in Klein's Seminars: a = Wedell, b = Zapolskaya, c = Chisholm Young, d = Bosworth, e = Palmie, f = Gernet, g = Afanasyeva, h = Lebedeva

[a] https://www.sub.uni-goettingen.de/sammlungen-historische-bestaende/alte-drucke-1501-1900/historische-vorlesungsverzeichnisse/

[b] Subtitled *Ship theory*. In winter 1899/1900 Hilbert offered a seminar on integral calculus.

[c] For the participation of the Privatdozent Max Abraham, see Klein (1921–1923), Vol. III, Appendix, p. 9.

[d] For the summer of 1900 Hilbert announced "Exercises on Surface Theory".

[e] For WS 1900/01 Hilbert announced: "Exercises on the Theory of Partial Differential Equations".

[f] Announcement: "Felix Klein with Dr Sommer, Lectures by Members on Selected Questions of Differential Geometry". Julius Sommer, a student of A. Brill, had habilitated in Göttingen in 1899 but had already been offered a professorship (Extraordinariat at the Agricultural University of Bonn) in 1901. Klein therefore ran the seminar alone and chose geodesy, a subject that had been included in the new Teaching qualification of Applied Mathematics as part of the examination regulations for teacher candidates in 1898.

[g] Karl Schwarzschild was appointed professor of astronomy in 1901.

[h] The official announcement for 1902/03 in the course catalogue was: Prof. Klein and Prof. Bohlmann, "Selected Questions in the Calculus of Probability". – Georg Bohlmann had been inspired by Klein to habilitate at Göttingen in 1894 and had been steered towards actuarial mathematics. In 1903 he joined an insurance company as chief mathematician.

[i] Official announcement: "In the mathematical-physical seminar, Professors Klein and Schwarzschild, assisted by Prof. Brendel (insurance seminar), conduct lectures by members on selected questions of probability theory." – Martin Brendel was extraordinary professor of theoretical astronomy and geodesy at Göttingen from 1898 to 1907. – See the section by Hans Fischer in this book.

[j] Official announcement: Klein & Schwarzschild. – Klein did indeed include the newly appointed Ludwig Prandtl and Carl Runge, as well as the aforementioned Woldemar Voigt and Hermann Theodor Simon, Professor of Applied Electricity, in the management.

[k] The course catalogue for 1905/1906 announced: (a) Electrical engineering again as in SS 1905; (b) Hilbert and Minkowski: Lectures on differential equations. – Klein organised the joint seminar before the start of the semester (see Tobies (2025, 455–460)) – In 1906/1907, Gustav Herglotz, who had habilitated under Klein in 1904, joined the administration.

[l] These courses allowed for less presentations by students. There were 51 students enrolled for 1905/1906, 32 for 1906, 25 for 1906/1907, and 22 for 1907.

[m] Emil Wiechert, Professor of Geophysics.

[n] Subtitled *Ship Theory and Dynamical Meteorology*.

[o] Klein had originally announced something different: "Introduction to recent mathematical literature".

[p] Felix Bernstein was Associate Professor of Actuarial Mathematics, Mathematical Statistics and Probability Theory from 1909. From 1911, actuarial mathematics became an optional subject in the applied mathematics section of the examination regulations for teacher training candidates.

International Congress of Mathematicians in Zurich in 1897, where she met Felix Klein.[107] Vera I. Schiff had also been a founding member of the aforementioned St Petersburg Mathematical Society in 1890, and she recommended twelve women, mostly her former students, for admission to the Society;[108] on 19 October 1894 Ljubov Nikolaevna Zapolskaya [Sapolsky, Sapolski] was elected a member. In the winter term 1895/1896 Zapolskaya attended Klein's courses on number theory and theory of the top; she also gave a lecture in the number theory seminar,[109] led by Klein and Hilbert. Her lecture topic led to her doctoral thesis, which she defended under Hilbert. She was followed by Nadjeschda von Gernet (see also Fig. 1.2) and Vera Lebedeva, who also completed their doctorates under Hilbert and went on to have distinguished careers.[110] Klein quickly recognised Tatyana A. Afanasyeva's talent for theoretical physics and mathematics and won her over (along with her Austrian husband) for the Encyklopädie article on statistical mechanics.[111] Klein's list of students shows that most foreign students attended lectures with him, even if they were doing their doctorates with Hilbert or others.[112]

1.3.3 The Seminars on Linear Differential Equations and Automorphic Functions

In his primarily application-oriented phase, Klein also continued to focus on the important fields of so-called pure mathematics. The seminar that he held for four semesters together with Hilbert and Minkowski is a fine expression of this (see Table 1.1). Klein made the main contribution to this; in 1905/1906 and 1907 only

[107] Four female mathematicians took part in Zurich; three of them were in close contact with Felix Klein: in addition to Vera I. Schiff, these were the American Charlotte A. Scott and the Dane Charlotte Wedell. Wedell formally obtained her doctorate in 1897 in Lausanne, Switzerland (inspired by Klein's former doctoral student Adolf Hurwitz at the Polytechnic Zurich, where there was still no right to award doctorates). Wedell had been studying with Klein in Göttingen since the winter semester of 1896/97. The fourth woman on the list of congress participants was the Italian Dr phil. Iginia Massarini, Rome, in the list of congress participants, at the same address as lawyer Camillo Massarini and Mrs Teresa Massarini, presumably her parents (see Rudio (1898, 72)). Iginia Massarini had completed her doctorate in Naples in 1887, supervised by Giuseppe Battaglini, and had been a member of the Circolo Matematico di Palermo since 1895 (for her dimly known biography see Carbone and Talamo (2010, 25–28)).

[108] See also A.I. Nazarov and G.I. Sinkevich 2018: https://arxiv.org/pdf/1812.03231.pdf.

[109] [PROTOKOLLE], vol. 12, pp. 8–17 (Zapolskaya's talk on the Theory of Higher Congruences, dated 12 February 1896). She attended Klein's courses and seminars every term until the summer term of 1898, cf. Tobies (2020). See also https://continuum-journal.ru/media/docs/articles/2021/3/12.pdf.

[110] Klein's and Hilbert's reviews of the women's doctoral theses are published in Tobies (1999); for the courses taken see Tobies (2020); for Lebedeva see also Poggendorff, vol. V, pp. 891–892.

[111] For other female mathematicians who studied under Klein or had important contacts with him, see Tobies (2020), and the section by June Barrow-Green and Brigitte Stenhouse in this book.

[112] Hilbert's first female doctoral student was Anne Lucy Bosworth from the United States, who graduated in 1900.

Klein lectured, as he had made significant results in the field himself. He also gave a few lectures in each of the other two terms.

For the summer term 1906, we read in Klein's handwriting: "Mathematical Seminar. Summer 1906 in connection with Hilbert and Minkowski, led by Klein." ([PROTOKOLLE], vol. 24, p. 1) For this seminar, Klein invited twelve people to give talks and took on two topics on works by Poincaré himself. Klein left it to Franz [František] Graf, who had already completed his doctorate in Prague,[113] to enter the details of his own lectures in [PROTOKOLLE] vol. 24, pp. 125–36. Graf's own lectures followed on from Klein's topics: "Group and uniformisation [...]" (vol. 24, pp. 137–46); "Monodromy group of hypergeometric functions, derived from Wirtinger's formula" (vol. 25, pp. 146–92). In addition to Graf and the Hungarian Alfréd Haar (see below), the foreign lecturers in the summer term of 1906 included Michael I. Akimoff [Akimov] from St. Petersburg, who had attended Klein's courses for four terms since 1904/1905.[114] Akimoff spoke "On the limiting cases of the Riemann *P*-function" ([PROTOKOLLE], vol. 24, pp. 51–56). In his lecture he referred, among other things, to the results of the doctoral theses of Klein's students Richard Olbricht and Friedrich Schilling.

The 1906/1907 term was mainly devoted to the topic of oscillation theorems. Klein had first stated his oscillation theorem in his article "Zur Theorie der Laméschen Functionen" [On the Theory of Lamé's Functions] (1890).[115] At the beginning Hilbert gave his only lecture of the four semesters. He used methods of the calculus of variations and said, among other things: "Finally, the development theorem, as Klein established it for the two-parameter problem, has a completely different content here; it can be derived using the tools of the calculus of variations [...]." Klein remarked in the discussion

> that in the oscillation theorem, as he obtained it from the automorphic functions, the boundary conditions to be used were neither the simplest homogeneous ones that had just been assumed, nor would they otherwise have been those that Hilbert had described [...]. These are always relations between the boundary values themselves and the values obtained by analytically continuing the particular solution from one end of the interval to the other.

In the discussion, Minkowski – who did not give a presentation of his own in the four semesters – asked for a geometric interpretation of the two-parameter development

[113] Graf studied in Göttingen for four semesters (1905/1906 to 1907); took part in all four seminars (in addition to attending some of Klein's lecture courses). See also Martina Bečvářová's Chap. 3.

[114] He later became a professor at the Institute of Theoretical Mechanics in St Petersburg. http://www.gff-lgi.spb.ru/teachers/leading_scientists/ls_akimov_m_i.htm. A number of other scientists from Russia who enrolled in Klein's courses, e.g. Efimov (1899/1900), could not be identified.

[115] *Göttinger Nachrichten* 4 (1890), pp. 85–95; reprinted in Klein (1921–1923), Vol. II, pp. 540–549.

theorem on the principal axes or normal problem.[116] Klein followed this up with another lecture of his own "On the Oscillation Theorem and Conformal Mapping" and other contributions to clarify, among other things, the relationship to Hilbert's approaches (vol. 25, pp. 12–34; 105). Paul Koebe, who recorded this lecture by Klein in the *Protocol book*, went on to talk about his own results, which enabled him to complete his habilitation in Göttingen, etc. The spectrum was completed by Ernst Hellinger's lecture on "The work of M. Bôcher on Klein's oscillation theorem" (vol. 25, pp. 88–104).[117] The lectures of Wilhelm Ihlenburg (vol. 24, pp. 101–17, vol. 25, pp. 217–27; 229–48) led to his doctoral dissertation (1909), supervised by Klein. Klein also inspired the Austrian Hermann Rothe to write his doctoral thesis. Rothe had written Klein's summary of the *oscillation theorem* in the [PROTOKOLLE] (vol. 25, pp. 249–61), on 6 March 1907, the last session of the 1906/1907 seminar. He then presented calculations requested by Klein (pp. 262–66) which became the subject of his dissertation, for which he received a doctorate from the Technische Hochschule (TH) in Vienna in 1909.[118] Rothe joined the DMV in 1910. Klein also won him as one of the authors for volume 3 (geometry) of the Encyklopädie. The Pole Antoni Łomnicki, who had graduated from the University of Lwów in 1903, followed Rothe in the seminar session with a similar topic, the numerical calculation of the parameter with accessors for $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \rho)$. He signed his name as Dr Antoni R. v. Łomnicki (vol. 25, pp. 267–71). He became an expert in applied mathematics, especially mathematical geography, and also probability theory (see Hans Fischer's contribution in this book).

The Austrian Robert König took up Hilbert's approach. His lecture topic (vol. 25, pp. 76–87) led to his dissertation *Oszillationseigenschaften der Eigenfunktionen der Integralgleichung mit definitem Kern und das Jacobische Kriterium der Variationsrechnung* [Oscillation properties of the eigenfunctions of the integral equation with definite kernel and the Jacobian criterion of the calculus of variations], which he defended with Hilbert in 1907.

Alfréd Haar, who would also graduate under Hilbert (1909),[119] spoke on *Riemann's problem* at three seminar sessions (vol. 25, pp. 106–21; 201–16). Haar

[116] Hilbert's seminar lecture and the discussion were summarized in the protocol book by Otto Toeplitz, who himself also spoke about algebraic analogies of the oscillation theorems, see in [PROTOKOLLE] vol. 25, citations p. 9 (Hilbert), and 10–11 (Klein); 11 (Minkowski); Toeplitz's lecture (ibid., pp. 65–75). – Report on Klein's discussion comment: "daß bei dem Oscillationstheorem, wie er es von den automorphen Funktionen aus erhalte, die zu benutzenden Randbedingungen weder die eben immer angenommenen einfachsten homogen seien, noch auch sonst unter den von Hilbert in seiner „2. Mitteilung" angeführten enthalten wäre[n]. Da handle es sich immer um Relationen zwischen den Randwerten selbst und den Werten, die man durch analytische Fortsetzung der betreffenden Particularlösung von einem Ende des Intervalls zum anderen erhält."

[117] The American Maxime Bôcher had completed his doctorate under Klein in 1891. Hellinger received his doctorate under Hilbert in 1907 and then became Klein's assistant for three semesters.

[118] Published on 1 December 1908 in *Monatshefte für Physik und Mathematik* 29 (1908) 258–88.

[119] Haar's doctoral thesis: "Zur Theorie der orthogonalen Funktionensysteme (Erste Mitteilung)". *Math.Ann.* 69 (1910) 33–71. He submitted his habilitation thesis in Göttingen as early as December

had already heard Klein's lecture on projective geometry in 1905/1906, attended the entire cycle of seminars and continued to attend Klein's seminar in 1909. During his time at Göttingen he acquired a broad knowledge, including Lie's theory of continuous groups, which was to become an important basis for his later work on an invariant measure on locally compact groups, now called the *Haar measure*.[120] Klein's inspiration can also be seen in the fact that Haar dealt with Chebyshev approximations and made important contributions to approximation theory; *Haar wavelet*, *Haar space*, and *Haar transform* are also named in his honor.

Dénes Kőnig attended courses by Klein, Hilbert and Minkowski from 1904/05 and is said to have been particularly influenced by the latter. In his lecture on 7 February 1907, Dénes Kőnig provided an overview of *The finite ternary groups*, including Klein's group G_{168}, and results by Valentiner, Wiman, Maschke (vol. 25, pp.193–200). Shortly afterwards, in 1907, he submitted his dissertation Elementary discussion of rotations and finite rotation group[s] of a space of many dimensions[121] in Budapest and achieved further results.[122]

A number of Hungarian mathematicians who studied in Göttingen at the beginning of the twentieth century became members of the DMV: Friedrich [Firgyes] Riesz, who is regarded as one of the founders of functional analysis, in 1905; Fejér and Goldziher in 1906; Haar and Kármán in 1909, Otto Szász in 1912; Dénes Kőnig and Marcel Riesz, the younger brother of F. Riesz, in 1922 (see Toepell (1991)). Marcel Riesz and Szász were among the authors of the analysis volume (2.3.2.) of the Encyklopädie.

1.3.4 Students from Hungary and Russia in Applied Courses

In the years following the 1898 examination regulations for teachers of the *Faculty of Applied Mathematics*,[123] Klein mainly concentrated on the new areas, which coincided with his work on the Mechanics Volume of the Encyklopädie. For this volume, Klein also engaged eleven foreign authors, including Hungarians (Kármán and Zemplén) and Russians (the above mentioned Tatjana Afanasyeva-Ehrenfest) who were studying with him. The applied mathematician Alexei Nikolajewitsch Krylov [Kriloff] from St Petersburg (see also Fig. 1.2) took part as well. He wrote on ship theory – Klein's seminar topic in 1899/1900.

1909: "Zur Theorie der orthogonalen Funktionensysteme (Zweite Mitteilung)". *Math.Ann.* 71 (1912) 38–53.

[120] Haar, A. (1932). "Der Massbegriff in der Theorie der kontinuierlichen Gruppen." (German trans. from the original Hungarian paper). In Karzel/Sörensen 1984, pp. 38–60.

[121] Original in Hungarian in *Mathematikai és Fizikai Lapok* 16, pp. 313–35, 373–90.

[122] See also Poggendorff, vol. V, pp. 650–51, and Franchella, Miriam (1997). "On the origins of Dénes Kőnig's infinity lemma." *Archive for the History of Exact Sciences* 51, pp. 3–27.

[123] For more details on this Faculty of Applied Mathematics see Tobies (2021, 445–50).

Horváth (2006, 246) reports that in the volumes of the *Math.Ann.* between 1900 and 1910, 34 articles (out of approx. 450) had Hungarian authors. Among them was Károly [Karl] Goldziher, who studied under Klein (and Hilbert) in 1900–1901.[124] He is regarded as a pioneer of applied mathematics in Hungary, influenced by Klein.[125] Most of Goldziher's later work was concerned with mathematical statistics, actuarial mathematics,[126] and the pedagogy of mathematics. When Klein visited Budapest in 1905 in connection with the above-mentioned Bolyai Prize Committee, he also gave a lecture on the ongoing teaching reform at the Budapest Mathematical-Physical Society. Goldziher sent him an article which he had published on Klein's lecture in a *Report of the* (Hungarian) *National Association of Secondary School Teachers*. Goldziher signed his letter with the words "Your student Dr K. Goldziher, who holds you in high esteem."[127]

Klein repeatedly emphasized the "famous" articles in the journal *Math.Ann.* written by Leopold Fejér on the approximation of Fourier series.[128] Fejér, who received his doctorate in Budapest in 1902, attended Klein's lectures and seminar in 1902/1903 (without giving a presentation). Fejér presented results in the Göttingen Mathematical Society,[129] and Klein won him as an author for his *Math.Ann.*[130] When the *Kgl. Gesellschaft der Wissenschaften zu Göttingen* [Royal Society of Sciences in Göttingen, later Academy] had vacancies to fill on July 1, 1917, Fejér was first on Klein's list of suggestions alongside Weyl.[131]

Some of the Russian mathematicians, like the Hungarian Goldziher, only attended Klein's lecture courses or were only enrolled in the seminar without giving a talk. The latter included Nikolai Nikolavech Saltykov, who has studied with Vladimir Steklov [Steckloff, see Fig. 1.4] at the University of Charkow and was enrolled as a 'non-student' at Klein's geodesy seminar of Summer 1901; in the same year he became associate professor of theoretical mechanics in Tomsk, in 1903 professor at the Polytechnic Institute in Kiev.[132]

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[124] Goldziher attended Klein's lectures on projective geometry and differential-integral calculus ([UBG]) Cod. MS. F. Klein 7 E: 194v, and 200v.

[125] His first paper published in *Math.Ann.* was Goldziher, K. (1905). "Beitrag zur Theorie der ersten Randwertaufgabe bei der allgemeinen linearen partiellen elliptischen Differentialgleichung 2. Ordnung." *Math.Ann.* 60 (1905) pp. 532–42 (submitted August 1904).

[126] Goldziher, Károly (1912). "Magyar biztosítottak halandósága." [Mortality of insured Hungarians] *Budapesti Szemle* 149, pp. 465–72. See also Erdélyi (2018), and Horváth (2006, 246, 576–77).

[127] [UBG] Cod. MS. F. Klein 34: Bl. 265, 265R (Goldziher's letter to Klein, dated 5 November 1905. Goldziher wrote his Report in German in *Berichte des Landesverbandes der Mittelschulprofessoren*; Goldziher wrote in the end of the letter to Klein, dated 5 November 1905: "Mit dem Ausdruck dankbarer Hochachtung verbleibe ich Ihr Sie hochschätzender Schüler Dr. K. Goldziher.").

[128] Klein (1928, 77,80,82,102). See also Horváth (2006, 249–253).

[129] See *Jahresbericht der DMV* 16 (1907), p. 454; p. 523.

[130] Fejér, L. (1904). "Untersuchungen über Fourierreihen." *Math.Ann.* 58, pp. 51–69 (dated Göttingen, January 1903); Fejér, L. (1907). "Über die Fouriersche Reihe." *Math.Ann.* 64, pp. 273–88.

[131] [UBG] Cod. MS. F. Klein 3 F: Bl. 5.

[132] Cf. on his further career: https://fr.wikipedia.org/wiki/Nikolai_Saltykow; and https://ihst.ru/projects/emigrants/saltykov.htm.

Gregor [Gregorij] A. Grusinzev [Grusinzeff] enrolled Klein's (application-oriented) courses on the Encyklopädie for two semesters (1902/03, 1903), and also took part in courses by Hilbert and Minkowski. Grusinzev was supported by D.M. Sintsov [Sinzow] – with whom Klein had good contacts (as mentioned above[133]) – and eventually obtained positions in Dnepropetrovsk and at the University of Kharkov.[134]

The Russian Alexei K. Vlasov [Wlassoff], who arrived on a scholarship in the autumn of 1903, had previously obtained a Master's degree in geometry in Moscow. He stayed for two semesters and in the summer of 1904 gave a seminar lecture on analytical methods of probability theory. It should be emphasized that Vlasov also lectured at the Göttingen Mathematical Society[135] and brought Göttingen teaching methods to Moscow. In 1906, together with the aforementioned Młodziejowski, Vlasov organised a seminar in Moscow (at the Higher Women's Courses and the University), modelled on the University of Göttingen.[136] He developed a series of textbooks on higher mathematics, including probability theory and its application to mathematical statistics.[137]

Antoni-Bonifatsi P. Psheborski (from a Polish family, see Chap. 2 by Danuta Ciesielska) is listed in the Göttingen student register as Anton Pscheborski from Charkow in 1904. He attended lectures by Klein, Hilbert and Minkowski, and expressed his enthusiasm in a report about Klein, the organisation of the course and the scientific working atmosphere. Psheborski taught as a professor in Kharkov and later in Warsaw. Among other things, he made important contributions to numerical mathematics; with the article "About some polynomials least deviating from zero on a given interval" (1913, Russ in the original) he extended a minimisation problem of V.A. Markov.[138]

Demetrius (Dmitri A.) Kryshanovsky from Odessa studied in Göttingen in 1904 and 1904/05. He also became a member of the DMV in 1913 (until 1937) and (as a professor in Odessa) translated Klein's Elementary Mathematics from a Higher Standpoint (Vol. I and II) into Russian (Moscow/Leningrad 1933 and 1934; 2nd edition 1987).[139]

[133] See also Demidov et al. (1994).

[134] See https://ru.wikipedia.org/wiki/Грузинцев,_Григорий_Алексеевич.

[135] On 7 June, 21 June and 2 August 1904 on mechanisms for constructing conic sections, and on a derivation of the theorems on the curvature of conic section systems, see *Jahresbericht der DMV* 13 (1904) pp. 485–87.

[136] https://ru.wikipedia.org/wiki/Власов,_Алексей_Константинович.

[137] "Теория вероятностей. Лекции" (1909) [Theorie der Wahrscheinlichkeiten. Lectures 1909]; "Теория вероятностей. С приложением к математической статистике. Курс лекций" (1916) [Probability Theory. With an Application to Mathematical Statistics. A Course of Lectures, 1916]. https://urss.ru/cgi-bin/db.pl?page=Avt&list=164223 – See also Hans Fischer's Chap. 15.

[138] See Steffens 2006: 167–75; 213, and https://uk.wikipedia.org/wiki/Пшеборський_Антоній-Боніфацій_Павлович.

[139] https://uk.wikipedia.org/wiki/Крижанівський_Дмитро_Антонович, https://archive.org/details/1_klejn_elementary_high, https://archive.org/details/2_klejn_elementary_high.

When Peter V. Woronetz came to Göttingen from Kiev in the winter semester of 1904/05, he had a Master's degree and had already published several papers that had been recognised in Göttingen. Now some of his articles, mainly on topics in mechanics, also appeared in the *Math.Ann.*[140] In 1908 he was appointed as a professor in Kiev.

There were also physicists, Alexander I. Tudorowski in 1905 and Vladimir D. Kuznetsov [Kusnetzoff] in 1909, the future engineer-scientist Nikolay M. Abramov in 1906 in Klein's courses.[141] There are others who have not yet been identified, such as Vladimir von Philippov [Philippow] (St Petersburg), who spoke about kink phenomena in Klein's Seminar of 1909 ([PROTOKOLLE], vol. 27, pp. 308–14). Stepan P. Timoshenko [Stephen Timoshenko] and the aforementioned Hungarians Kármán and Alfréd Haar also gave lectures in this seminar. Timoshenko in his autobiography gratefully recalled Klein's mathematical suggestion for his beam theory.[142]

Kármán, who had a degree in mechanical engineering from the Josephs University of Technical Sciences (famous for mathematics mentioned above), attended Klein's courses since 1906/1907 and took part in the seminar on hydrodynamics in 1907/1908 (Klein had Prandtl, Runge and Wiechert involved in the management of the seminar).[143] When Kármán submitted his dissertation in Göttingen on 12 July 1908, Prandtl was the main reviewer and judged opus eximium. In the oral doctoral examination, Kármán chose Applied Physics as his main subject and Mathematics and (Theoretical) Physics as his minor subjects.[144] Klein examined in mathematics and also recommended the habilitation. Thanks to Klein's inspiration,[145] Kármán found the decisive results for his famous vortex street. He presented the development stages of his results at the Göttingen Mathematical Society (on 21 Februar, 4 July

[140] Woronetz submitted his first paper "Über das Problem der Bewegung von vier Massenpunkten unter dem Einfluss von inneren Kräften" in November 1905 from Kiev, which was published in *Math.Ann.* 63 (1907) pp. 387–412. See also Poggendorff, vol. V, p. 1392.

[141] The author would like to thank Galina Sinkevich for biographical information on these scientists, see also Kalinin et al. (2024).

[142] See Tobies (2021, 471).

[143] The Hungarian Aladár Visnya, who already had a doctorate and had published two mathematical papers (on group theory) in Hungarian, was also involved in the 1907/1908 and 1908 seminars (([PROTOKOLLE])) vol. 27. He subsequently worked primarily as a teacher. See https://hu.wikipedia.org/wiki/Visnya_Alad%C3%A1r.

[144] [UBG] Phil. Prom. 141 Nr. 15.

[145] See also Klein, F. (1909). "Über die Bildung von Wirbeln in reibungslosen Flüssigkeiten." *Zeitschrift für Mathematik und Physik* 58 (1910), reprinted in Klein (1921–1923), Vol. II, pp. 710–713.

and 1 August 1911), and Klein submitted his work to the Royal Society of Sciences in Göttingen.[146]

Győző Zemplén, who has enjoyed a good education in Budapest, came to Göttingen in the autumn of 1904,[147] where he developed his theory of shock waves. Klein immediately recognised the importance of the results, as he had already given a course on hydrodynamics, which was also available to the students in the above-mentioned *Lesezimmer* (a study room for students with a library, as well as with the elaborations of Klein's older lecture courses). Klein asked him to write a chapter for the Encyklopädie on the discontinuous motions of fluids. Zemplén gave a lecture on this topic at the Göttingen Mathematical Society as early as 21 February 1905[148] and composed an article for *Math.Ann.* He then married and continued his studies in Paris.[149]

The list of members of the Göttingen Mathematical Society for the winter term 1910/11 shows 31 people, 14 of whom were foreigners, including one woman. The Hungarians involved (Haar, v. Kármán, M. Riesz) and those from what was then Tsarist Russia should be mentioned here: Miss Theodosia [Feodosia] Tarnarider from Kherson[150]; Alexeij L. Lavrentjev[151] from Kazan; Vladimir Kh. Davatz [da Vatz] and Michail N. Martschewsky from Kharkov. Felix Klein dominated the programme of this Society, which he had founded in 1892, even as an emeritus

[146] *Jahresbericht der DMV* 20 (1911) pp. 63; 144; 199 (K.'s presentations in the Göttingen Math. Society). Kármán, Th. (1911). "Über den Mechanismus des Widerstandes, den ein bewegter Körper in einer Flüssigkeit erfährt". (Submitted by F. Klein, 14 Sept. 1911). *Nachrichten von der Kgl. Gesellschaft der Wissenschaften zu Göttingen, math.-physikal. Klasse*, 1911, pp. 509–17; and 1912, pp. 547–56 (the second part was submitted by C. Runge on 23 December 1911; at this time, Klein was in a sanatorium). For Th. v. Kármán see also Reinhard Siegmund-Schultze's section in this book, and in Horváth (2006, 374–382).

[147] Zemplén gave lectures on 30 November, 7 December 1904, and 11 January 1905 ([PROTOKOLLE]) vol. 21.

[148] *Jahresbericht der DMV* 14 (1905): 201 (Report of Zemplén's lecture at the Göttingen Math. Society). The author would like to thank R. Siegmund-Schultze for this information.

[149] Zemplén, G. (1905). "Kriterien für die physikalische Bedeutung der unstetigen Lösungen der hydrodynamischen Bewegungsgleichungen." *Math.Ann.* 61, pp. 437–49 (dated in Paris, April 1905, already referring to his forthcoming Encyklopädie article (vol. 4.3, pp. 281–323); see also (Kovács 2004).

[150] Th. Tarnarider had studied in Kharkov or Dorpat and in Paris, where she obtained a recognised position at the *Bureau International des Poids et Mesures*. She translated a doctoral thesis from the University of Dorpat (1893) for the *Bulletin of the Société de Mathématique de France*, vol 38 (1910), p. 5; and she generalised the approximation methods of Sergei N. Bernstein (1880–1968), who referred to her as his student; Bernstein had also studied in Göttingen and attended Klein's Encyklopädie lecture course there in 1902/03 ([UBG] Cod. MS F. Klein 7 E: 209v). Tarnarider published her results, submitted by Picard, in *Comptes Rendus de l'Académie des sciences* 156 (1913) pp. 672–75, and 159 (1914) pp. 294–96. The author would like to thank D. Ciesielska for the biographical information.

[151] A.L. Lavrentjev was a professor of mechanics at the University of Kazan. In 1910/11, he travelled to Göttingen with his 10-year-old son Mikhail A. Lavrentjev, who later became an important mathematician. The son attended a school during this time. https://novosibirsk-history.ru/projects/lichnosti/mikhail_alekseevich_lavrentev/.

professor. When Pavel S. Aleksandrov and Pavel S. Urysohn first came to Göttingen in 1923, Nikolai N. Luzin[152] had given them a letter of recommendation to Klein (not to Hilbert). The letter contained a request to Klein to admit these mathematicians (they were to become famous for their work in topology) to the Göttingen Mathematical Society.[153]

Further examples show Klein's far-reaching impact in the areas of applications of mathematics. In 2015, a group led by William F. Baker elaborated a design tool for plane trusses using an extended Airy stress function based on Klein and his doctoral student Karl Wieghardt.[154] Klein's understanding of approximation mathematics enabled him to identify open problems, including what is now known as the Painlevé-Klein problem in the theory of friction.[155] Helmut Neunzert, the founder of the new study course of *Technomathematics* in Kaiserslautern, put the relationship between mathematics and technology on a modern footing around 45 years ago. He recently confirmed that he was inspired to do so by Klein's combination of pure and applied mathematics.[156]

Klein's extensive network helped him to spread the knowledge. This monograph will not be the end of the story. In other words, there are still unanswered questions.[157]

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[UBG]. *Niedersächsische Staats- und Universitätsbibliothek, Handschriftenabteilung. Nachlass Felix Klein: Cod. MS. F. Klein*.

[AdW Wien] Archive of the Austrian Academy of Sciences at Vienna. Files on Wilhelm Wirtinger.

[152] From 1910 to 1914, Luzin had mainly studied analytical number theory in Göttingen under Edmund Landau.

[153] The author received the letter from Martin Kneser (1928–2004), Göttingen, cf. also Tobies (1990).

[154] Klein, F.; Wieghardt, K. (1905). "Über Spannungsflächen und reziproke Diagramme mit besonderer Berücksichtigung der Maxwellschen Arbeiten." *Archiv der Mathematik und Physik* (3. Reihe) 8, pp. 1–10, 95–119. For Baker et al. see Kurrer (2018, 960–961).

[155] See Tobies (2021, 471–472), and Le Xuan Anh (2003). *Dynamics of Mechnical Systems with Coulomb Friction*. Berlin: Springer, pp 70, 141, 176.

[156] Helmut Neunzert to the author, on 29 July 2023.

[157] For example, it is worth investigating Klein's contact with the Swede Helge von Koch in more detail; he is on the list (Fig. 1.2). Koch is considered a co-founder of functional analysis; he described one of the earliest fractal curves (the famous fractal known as the Koch snowflake). He published his lecture from Paris (1900) "Ueber die Riemann'sche Primzahlfunction" in *Math.Ann.* 55 (1902) 441–64. – It is not surprising that Klein (President of the IMUK formed in 1908) was able to win him over as a Swedish delegate. See also Furinghetti and Giacardi (2022), and Siegmund-Schultze (2018).

[ARCHIV ST. PETERSBURG] Archive of the Academy of Science in St. Peters-burg. Fond 2, 810 (Election ptoposal for F. Klein, L. Fuchs, G. Darboux); A.A. Markov Fond 173, Opis 1, Nr. 40 (Klein's letters to Markov).

[BBF] Archiv der Bibliothek für bildungsgeschichtliche Forschung, Berlin: Personalbögen [Personnel files of Prussian school teachers].

[OSLO] Nasjonalbiblioteket in Oslo, Norway, Estate o fSophus Lie I (Klein' sletters to Lie).

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